Technical Appendices on "Understanding the Gains from Wage Flexibility in a Currency Union: The Fiscal Policy Connection"

OKANO, Eiji

eiji_okano@econ.nagoya-cu.ac.jp

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Chapter 1 Distorted Steady State Model

1 The Model

1.1 Distorted Steady State Model

1.1.1 Households in the SOE

Households' optimization problem in the SOE is given by:

$$\max_{C_t, C_{t+1}, D_{t+1}^n} \sum_{t=0}^{\infty} \beta^t U_t ,$$

s.t.

$$U_t \equiv \left(\ln C_t - \frac{1}{1+\varphi} \int_0^1 N_t(j)^{1+\varphi} dj \right) Z_t ,$$

$$D_t^n + \int_0^1 W_t(j) N_t(j) dj + PR_t^n + TR_t \geq P_t C_t + \mathrm{E}_t\left(Q_{t,t+1} D_{t+1}^n \right).$$

The Lagrangean is given by:

$$L \equiv \beta^t \left(\ln C_t - \frac{1}{1+\varphi} \int_0^1 N_t(j)^{1+\varphi} dj \right) Z_t + \beta^{t+1} \left(\ln C_{t+1} - \frac{1}{1+\varphi} \int_0^1 N_{t+1}(j)^{1+\varphi} dj \right) Z_{t+1} + \cdots$$

$$+\lambda_t \beta^t \left[D_t^n + \int_0^1 W_t(j) N_t(j) dj + PR_t^n - P_t C_t - \frac{D_{t+1}^n}{R_t} \right]$$

$$+\lambda_{t+1} \beta^{t+1} \left[\begin{array}{l} D_{t+1}^n + \int_0^1 W_{t+1}(j) N_{t+1}(j) dj + PR_{H,t+1}^n - P_{t+1} C_{t+1} \\ - \frac{D_{t+2}^n}{R_{t+1}} \end{array} \right] ,$$

$$+\cdots$$

with $Q_{t,t+1} = R_t^{-1}$ and $R_t \equiv 1 + r_t$.

FONCs are given by:

$$\frac{\partial L}{\partial C_t} = \beta^t \frac{1}{C_t} Z_t - \beta^t \lambda_t P_t = 0 ,$$

$$\frac{\partial L}{\partial C_{t+1}} = \beta^{t+1} \frac{1}{C_{t+1}} Z_{t+1} - \beta^{t+1} \lambda_{t+1} P_{t+1} = 0 ,$$

$$\frac{\partial L}{\partial D_{t+1}^n} = -\lambda_t \frac{1}{R_t} \beta^t + \lambda_{t+1} \beta^{t+1} = 0 ,$$

which can be rewritten as:

$\lambda_t = \frac{1}{P_t C_t} Z_t$, (1-1-1)

$\lambda_{t+1} = \frac{1}{P_{t+1} C_{t+1}} Z_{t+1}$, (1-1-2)

$\lambda_t = \beta \lambda_{t+1} R_t$, (1-1-4)

Combing Eqs.(1-1-1), (1-1-2) and (1-1-4) yields:

$$\frac{1}{P_t C_t} Z_t = \beta \frac{1}{P_{t+1} C_{t+1}} R_t Z_{t+1} ,$$

which can be rewritten as:

$\beta \mathrm{E}_t \left(\frac{P_t C_t}{P_{t+1} C_{t+1}} \frac{Z_{t+1}}{Z_t} \right) = \frac{1}{R_t}$. (1-1-7) [(4) in the text]

Combining Eqs.(1-1-1) and (1-1-6) yields:

1.1.3 International Risk sharing Condition

Eqs.(1-1-7) can be rewritten as:

$$\beta^{-1} = R_t\left(\frac{P_t C_t}{P_{t+1}C_{t+1}}\frac{Z_{t+1}}{Z_t}\right).$$

In the rest of the world (ROW), there might be the same condition:

$$\beta^{-1} = R_t^*\left(\frac{P_t^* C_t^*}{P_{t+1}^* C_{t+1}^*}\frac{Z_{t+1}^*}{Z_t^*}\right),$$

with $R_t^* \equiv 1 + r_t^*$.

Combining the previous expressions each other yields:

$$R_t\left(\frac{P_t C_t}{P_{t+1}C_{t+1}}\frac{Z_{t+1}}{Z_t}\right) = R_t^*\left(\frac{P_t^* C_t^*}{P_{t+1}^* C_{t+1}^*}\frac{Z_{t+1}^*}{Z_t^*}\right).$$

Dividing both sides of the previous expression by R_t yields:

$$\frac{P_t C_t}{P_{t+1}C_{t+1}} = \frac{R_t^*}{R_t}\left(\frac{P_t^* C_t^*}{P_{t+1}^* C_{t+1}^*}\right)\frac{Z_t}{Z_{t+1}}\frac{Z_{t+1}^*}{Z_t^*}.$$

Plugging the UIP $R_t / R_t^* = \mathrm{E}_t\left(E_{t+1}\right) / E_t$ into the previous expression yields:

$$\frac{P_t C_t}{P_{t+1}C_{t+1}} = \frac{E_t}{E_{t+1}}\left(\frac{P_t^* C_t^*}{P_{t+1}^* C_{t+1}^*}\right)\frac{Z_t}{Z_{t+1}}\frac{Z_{t+1}^*}{Z_t^*},$$

which can be rewritten as:

$$\begin{aligned}\frac{C_t}{C_{t+1}} &= \frac{E_t P_t^*}{P_t}\frac{P_{t+1}}{E_{t+1}P_{t+1}^*}\frac{C_t^*}{C_{t+1}^*}\frac{Z_t}{Z_{t+1}}\frac{Z_{t+1}^*}{Z_t^*} \\ &= \frac{Q_t}{Q_{t+1}}\frac{C_t^*}{C_{t+1}^*}\frac{Z_t}{Z_{t+1}}\frac{Z_{t+1}^*}{Z_t^*}\end{aligned},$$

where we use the definition of the real exchange rate. Further, the previous expression can be rewritten as:

$$\frac{C_{t+1}}{C_t} = \frac{Q_{t+1}}{Q_t}\frac{C_{t+1}^*}{C_t^*}\frac{Z_{t+1}}{Z_t}\frac{Z_t^*}{Z_{t+1}^*},$$

by raising both sides to －1th power. Note that we assume for all *t*.
In the period －1, the previous expression is given by:

$$\frac{C_0}{C_{-1}}=\frac{Q_0}{Q_{-1}}\frac{C_0^*}{C_{-1}^*}\frac{Z_0}{Z_{-1}}\frac{Z_{-1}^*}{Z_0^*},$$

which can be rewritten as:

$$C_0=Q_0C_0^*\frac{Z_0}{Z_0^*}\left(\frac{1}{Q_{-1}}\frac{C_{-1}}{C_{-1}^*}\frac{Z_{-1}^*}{Z_{-1}}\right).$$

Let define $\vartheta\equiv\frac{1}{Q_{-1}}\frac{C_{-1}}{C_{-1}^*}\frac{Z_{-1}^*}{Z_{-1}}$ as an initial condition. Then the previous expression can be generalized as follows:

$C_t=Q_tC_t^*\frac{Z_t}{Z_t^*}\vartheta$. (1-1-16) [(9) in the text]

We assume that the discount factor shifter for foreign households follows:

$Z_t^*=Z_{1,t}^*Z_{2,t}^*$. (1-1-17)

Plugging Eq.(1-1-17) into Eq.(1-1-15) yields:

$\beta\mathrm{E}_t\left(\frac{C_t^*}{C_{t+1}^*}\frac{Z_{1,t+1}^*Z_{2,t+1}^*}{Z_{1,t}^*Z_{2,t}^*}\right)=\frac{1}{R_t^*}$ (1-1-18)

World interest rate remains unchanged in response to $Z_{1,t}^*$. However, the world interest rate is assumed to respond to $Z_{2,t}^*$ by adjusting the real interest rate to keep C_t^*.

Latter assumption implies as follows:

$\beta\mathrm{E}_t\left(\frac{Z_{2,t+1}^*}{Z_{2,t}^*}\right)=\frac{1}{R_t^*}$. (1-1-19)

Plugging Eq.(1-1-19) into Eq.(1-1-18) yields:

$$\frac{C_t^*}{Z_{1,t}^*}=\beta\mathrm{E}_t\left(\frac{C_{t+1}^*}{Z_{1,t+1}^*}\right).$$

In period 0, the previous expression can be rewritten as:

$$\vartheta_1=\frac{C_0^*}{Z_{1,0}^*}$$

with $\vartheta_1 \equiv \beta^{-1} \frac{C^*_{-1}}{Z^*_{1,-1}}$. Then the previous expression can be generalized as follows:

$C^*_t = Z^*_{1,t} \vartheta_1^{-1}$. (1-1-20). [(12) in the text]

Plugging Eq.(1-1-20) into Eq.(1-1-16) yields:

$C_t = Q_t \frac{Z_t}{Z^*_{2,t}}$. (1-1-21) [(13) in the text]

1.1.8 Optimal Allocation of Goods

Let define consumption indices as follows:

$$C_t \equiv \left[(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}, \text{ (1-1-27)}$$

with $C_{H,t} \equiv \left[\int_0^1 C_{H,t}(i)^{\frac{\varepsilon_p-1}{\varepsilon_p}} di \right]^{\frac{\varepsilon_p}{\varepsilon_p-1}}$.

We assume $EX_t \equiv \left[\int_0^1 EX_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$.

By solving cost-minimization problems for households, we have optimal allocation of expenditures as follows:

$$C_{H,t}(i) = \left(\frac{P_t(i)}{P_{H,t}} \right)^{-\varepsilon} C_{H,t}, \text{ (1-1-31)}$$

And:

$$EX_t(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\varepsilon_p} EX_t, \text{ (1-1-32)}$$

with:

$$P_{H,t} \equiv \left[\int_0^{\tilde{\alpha}} P_{H,t}(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}.$$

Now, we get total demand for goods produced in the SOE and the ROW. Optimazation problem is given by:

$\max_{C_{H,t}, C_{F,t}} C_t$,

s.t.

Eq.(1-1-27) and $\Xi_t - \left(P_{H,t}C_{H,t} + P_{F,t}C_{F,t}\right) = 0$.

The Lagrangean is given by

$$L \equiv \left[(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} + \lambda\left(\Xi_t - P_{H,t}C_{H,t} - P_{F,t}C_{F,t}\right).$$

The FONCs is given by:

$$\begin{aligned}\frac{\partial L}{\partial C_{H,t}} &= \frac{\eta}{\eta-1}\left[(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}-1} (1-\upsilon)^{\frac{1}{\eta}} \frac{\eta-1}{\eta} C_{H,t}^{\frac{\eta-1}{\eta}-1} - \lambda P_{H,t} \\ &= \left[\upsilon^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + (1-\upsilon)^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}-1} (1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} - \lambda P_{H,t} \\ &= 0\end{aligned}$$

$$\begin{aligned}\frac{\partial L}{\partial C_{F,t}} &= \frac{\eta}{\eta-1}\left[(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}-1} \upsilon^{\frac{1}{\eta}} \frac{\eta-1}{\eta} C_{F,t}^{\frac{\eta-1}{\eta}-1} - \lambda P_{F,t} \\ &= \left[\upsilon^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + (1-\upsilon)^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}-1} \upsilon^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}-1} - \lambda P_{F,t} \\ &= 0\end{aligned}$$

These previous expressions can be rewritten as:

$$\left[(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}-1} (1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}-1} = \lambda P_{H,t},$$

$$\left[(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}-1} \upsilon^{\frac{1}{\eta}} C_{F,t}^{\frac{\eta-1}{\eta}-1} = \lambda P_{F,t}.$$

Combining these expression yields:

$$\frac{(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{-\frac{1}{\eta}}}{\upsilon^{\frac{1}{\eta}} C_{F,t}^{-\frac{1}{\eta}}} = \frac{P_{H,t}}{P_{F,t}},$$

which can be rewritten as:

$$C_{F,t} = \frac{\upsilon}{1-\upsilon}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta} C_{H,t} \text{. (1-35)}$$

Plugging Eq.(1-1-35) into Eq.(1-1-27) yields:

$$\begin{aligned}
C_t &= \left\{(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}}\left[\frac{\upsilon}{1-\upsilon}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta} C_{H,t}\right]^{\frac{\eta-1}{\eta}}\right\}^{\frac{\eta}{\eta-1}} \\
&= \left[(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \upsilon^{\frac{1}{\eta}}\left(\frac{\upsilon}{1-\upsilon}\right)^{\frac{\eta-1}{\eta}}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta-1} C_{H,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \\
&= \left[(1-\upsilon)^{\frac{1}{\eta}} C_{H,t}^{\frac{\eta-1}{\eta}} + \upsilon(1-\upsilon)^{-\frac{\eta-1}{\eta}}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta-1} C_{H,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} \\
&= \left\{\left[(1-\upsilon)^{\frac{1}{\eta}} + \upsilon(1-\upsilon)^{-\frac{\eta-1}{\eta}}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta-1}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta-1}\right] C_{H,t}^{\frac{\eta-1}{\eta}}\right\}^{\frac{\eta}{\eta-1}} \\
&= \left\{\left[(1-\upsilon)^{\frac{1}{\eta}} + \upsilon(1-\upsilon)^{-\frac{\eta-1}{\eta}}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta-1}\right] C_{H,t}^{\frac{\eta-1}{\eta}}\right\}^{\frac{\eta}{\eta-1}} \\
&= \left[(1-\upsilon)^{\frac{1}{\eta}} + \upsilon(1-\upsilon)^{-\frac{\eta-1}{\eta}}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta-1}\right]^{\frac{\eta}{\eta-1}} C_{H,t} \\
&= \left\{\left[1 + (1-\upsilon)^{-\frac{1}{\eta}}\upsilon(1-\upsilon)^{-\frac{\eta-1}{\eta}}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta-1}\right](1-\upsilon)^{\frac{1}{\eta}}\right\}^{\frac{\eta}{\eta-1}} C_{H,t} \\
&= \left\{\left[1 + \frac{\upsilon}{1-\upsilon}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta-1}\right](1-\upsilon)^{\frac{1}{\eta}}\right\}^{\frac{\eta}{\eta-1}} C_{H,t} \\
&= \left[1 + \frac{\upsilon}{1-\upsilon}\left(\frac{P_{H,t}}{P_{F,t}}\right)^{\eta-1}\right]^{\frac{\eta}{\eta-1}} (1-\upsilon)^{\frac{1}{\eta-1}} C_{H,t}
\end{aligned} \text{. (1-1-36)}$$

The definition of the PPI in the SOE is given by:

$$P_t \equiv \left[(1-\upsilon)P_{H,t}^{1-\eta} + \upsilon P_{F,t}^{1-\eta}\right]^{\frac{1}{1-\eta}} \text{, (1-1-37)}$$

which can be rewritten as:

$P_t^{1-\eta} = (1-\upsilon)P_{H,t}^{1-\eta} + \upsilon P_{F,t}^{1-\eta}$. (1-1-38)

Further, Eq.(1-38) can be rewritten as:

$P_{F,t}^{1-\eta} = \frac{1}{\upsilon}P_t^{1-\eta} - \frac{1-\upsilon}{\upsilon}P_{H,t}^{1-\eta}$.

Plugging the previous expression into Eq.(1-1-36) yields:

$$
\begin{aligned}
C_t &= \left[1+\frac{\upsilon}{1-\upsilon}P_{H,t}^{\eta-1}\left(\frac{1}{\upsilon}P_t^{1-\eta}-\frac{1-\upsilon}{\upsilon}P_{H,t}^{1-\eta}\right)\right]^{\frac{\eta}{\eta-1}}(1-\upsilon)^{\frac{1}{\eta-1}}C_{H,t} \\
&= \left[1+\left(\frac{1}{1-\upsilon}P_{H,t}^{\eta-1}P_t^{1-\eta}-P_{H,t}^{\eta-1}P_{H,t}^{1-\eta}\right)\right]^{\frac{\eta}{\eta-1}}(1-\upsilon)^{\frac{1}{\eta-1}}C_{H,t} \\
&= \left\{1+\left[\frac{1}{1-\upsilon}\left(\frac{P_{H,t}}{P_t}\right)^{\eta-1}-1\right]\right\}^{\frac{\eta}{\eta-1}}(1-\upsilon)^{\frac{1}{\eta-1}}C_{H,t} \\
&= \left\{\left[\frac{1}{1-\upsilon}\left(\frac{P_{H,t}}{P_t}\right)^{\eta-1}\right]\right\}^{\frac{\eta}{\eta-1}}(1-\upsilon)^{\frac{1}{\eta-1}}C_{H,t} \\
&= \left(\frac{P_{H,t}}{P_t}\right)^{\eta}(1-\upsilon)^{-\frac{\eta}{\eta-1}}(1-\upsilon)^{\frac{1}{\eta-1}}C_{H,t} \\
&= \left(\frac{P_{H,t}}{P_t}\right)^{\eta}\frac{1}{1-\upsilon}C_{H,t}
\end{aligned}
$$

Then we have:

$C_{H,t} = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta}C_t$. (1-1-39)

We assume that:

$EX_t = \upsilon\left(\frac{P_{H,t}^*}{P_t^*}\right)^{-\eta}C_t^*$, (1-1-43).

1.1.9 Market Clearing Condition

The market clearing conditions in the SOE is given by:

$Y_t(i) = C_t(i) + EX_t(i)$. (1-1-68)

Let define $Y_t \equiv \left[\int_0^1 Y_t(i)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} di \right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}}$. Combining them with the definitions of the PPI indices yields:

$Y_t(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\varepsilon_p} Y_t$. (1-1-70)

Eq.(1-1-43) can be rewritten as:

$$
\begin{aligned}
EX_t &= \upsilon \left(\frac{P_{H,t}^*}{P_t^*} \right)^{-\eta} C_t^* \\
&= \upsilon \left(\frac{P_{H,t} / E_t}{P_{F,t}^*} \right)^{-\eta} C_t^* \\
&= \upsilon \left(\frac{P_{H,t} / E_t}{P_{F,t} / E_t} \right)^{-\eta} C_t^* \text{ , (1-1-43).} \\
&= \upsilon \left(\frac{P_{H,t}}{P_{F,t}} \right)^{-\eta} C_t^* \\
&= \upsilon S^{\eta} C_t^*
\end{aligned}
$$

Plugging Eqs. (1-1-31), (1-1-32), (1-1-39),(1-1-70), (1-1-72) and (1-1-73) into Eq.(1-1-68) yields:

$$\left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\varepsilon_p} Y_t = \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\varepsilon_p} \left[\left(\frac{P_{H,t}}{P_t} \right)^{-\eta} (1-\upsilon) C_t + S_t^{\eta} \upsilon C_t^* \right]$$

which can be rewritten as:

$Y_t = (1-\upsilon) \left(\frac{P_{H,t}}{P_t} \right)^{-\eta} C_t + \upsilon S_t^{\eta} C_t^*$. (1-1-76)

We assume that the global market clearing condition is given by:

$Y_t^* = C_t^*$. (1-1-77)

because $1-\upsilon = \alpha$ and $\upsilon^* = 0$.

1.1.10 Firms

Production function is given by:

$Y_t(i) = A_t N_t(i)$,

with

$$N_t(i) \equiv \left[\int_0^1 N_t(i,j)^{1-\frac{1}{\varepsilon_w}} dj\right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}}$$

Plugging the production function into Eq.(1-1-70) yields:

$$N_t(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} \frac{Y_t}{A_t}. \text{ (1-1-78).}$$

Integrating the previous expression yields:

$$\begin{aligned}\int_0^1 N_t(i) di &= \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} \frac{Y_t}{A_t} di \\ &= \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di \frac{Y_t}{A_t}\end{aligned},$$

which can be rewritten as:

$$N_t = \frac{Y_t \Delta_{p,t}}{A_t}, \text{ (1-1-80)}$$

with $N_t = \int_0^1 N_t(i) di$ and:

$$\Delta_{p,t} = \int_0^1 \left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} dh. \text{ (1-1-81)}$$

The FONC for firms is given by:

$$\tilde{P}_{H,t} = \frac{\frac{\varepsilon_p}{\varepsilon_p - 1} \sum_{k=0}^{\infty} (\beta\theta)^k \mathrm{E}_t \left[(P_{t+k} C_{t+k})^{-1} \tilde{Y}_{t+k} P_{H,t+k} MC_{t+k}\right]}{\sum_{k=0}^{\infty} (\beta\theta)^k \mathrm{E}_t \left[(P_{t+k} C_{t+k})^{-1} \tilde{Y}_{t+k}\right]}, \text{ (1-1-82)}$$

with $\tilde{Y}_{t+k} \equiv \left(\frac{\tilde{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon_p} Y_{t+k}$ and:

$$MC_t \equiv \frac{W_t}{(1-\tau_t) P_{H,t} A_t}. \text{ (1-1-83)}$$

1.1.11 Optimal Wage setting

Consider a household resetting its wage in period *t* to maximize:

$$\tilde{U}_H \equiv \mathrm{E}_t\left(\sum_{k=0}^{\infty}\beta^t \tilde{U}_{H,t+k}\right). \text{ (1-1-84)}$$

with:

$$\tilde{U}_{H,t+k} \equiv \left[\ln C_{t+k} - \frac{1}{1+\varphi}\int_0^1 N_{t+k|t}(j)^{1+\varphi}\, dj\right] Z_t . \text{ (1-1-85)}$$

The maximization of Eq.(1-1-84) is the subject to the sequence of labor demand schedules and a sequences of budget constraints of the form:

$$N_{t+k|t}(j) = \left(\frac{\tilde{W}_t(j)}{W_{t+k}}\right)^{-\varepsilon_W} N_{t+k}, \text{ (1-1-86)}$$

$$D_t^n + \int_0^1 \tilde{W}_t(j) N_{t+k|t}(j)dj + PR_t^n + TR_t \geq P_t C_t + \mathrm{E}_t\left(Q_{t,t+1}D_{t+1}^n\right). \text{ (1-1-87)}$$

We now make explicit that the households can pool labor income risk through government debt. Each household j reoptimizing the wage at a given time t will choose the same optimal wage. Because of this, we can abstract from index j on Eqs. (1-1-85), (1-1-86) and (1-1-87). Then these are given by:

$$\tilde{U}_{H,t+k} \equiv \left(\ln C_{t+k} - \frac{1}{1+\psi} N_{t+k|t}^{1+\varphi}\right) Z_t, \text{ (1-1-88)}$$

$$N_{t+k|t} = \left(\frac{\tilde{W}_t}{W_{t+k}}\right)^{-\varepsilon_W} N_{t+k}, \text{ (1-1-89)}$$

$$D_t^n + \tilde{W}_t N_{t+k|t} + PR_{H,t}^n \geq P_t C_t + \mathrm{E}_t\left(Q_{t,t+1}D_{t+1}^n\right). \text{ (1-1-90)}$$

Because of Eqs.(1-1-88) and (1-1-90), the Lagrangean is given by:

$$L \equiv \sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left[\ln C_{t+k} - \frac{1}{1+\varphi}N_{t+K|t}^{1+\varphi} + \lambda_{t+k|t}\left(\tilde{W}_t N_{t+k|t} - P_{t+k}C_{t+k}\right)\right],$$

which can be rewritten as:

$$L \equiv \left(\ln C_t - \frac{1}{1+\varphi}N_{t|t}^{1+\varphi}\right)Z_t + \lambda_{t|t}\left(\tilde{W}_t N_{H,t|t} - P_t C_t\right)$$
$$+\beta\theta_W \mathrm{E}_t\left[\left(\ln C_{t+1} - \frac{1}{1+\varphi}N_{t+1|t}^{1+\varphi}\right)Z_{t+1} + \lambda_{t+1|t}\left(\tilde{W}_t N_{H,t+1|t} - P_{t+1}C_{t+1}\right)\right] + \cdots$$

The FONC is given by:

$$
\begin{aligned}
&-N_{t|t}^{\varphi} Z_t \frac{\partial N_{t|t}}{\partial \tilde{W}_t} + \lambda_{t|t}\left(N_{t|t} + \tilde{W}_{H,t}\frac{\partial N_{t|t}}{\partial \tilde{W}_t}\right) \\
&+\beta\theta_W \mathrm{E}_t\left[-N_{t+1|t}^{\varphi} Z_{t+1}\frac{\partial N_{t+1|t}}{\partial \tilde{W}_{H,t}} + \lambda_{t+1|t}\left(N_{t+1|t} + \tilde{W}_t \frac{\partial N_{t+1|t}}{\partial \tilde{W}_t}\right)\right] + \cdots = 0
\end{aligned}.
$$

The compact form of the previous expression is given by:

$$
\sum_{k=0}^{\infty}(\beta\theta_W)^k\left[-N_{t+k|t}^{\varphi} Z_{t+k}\frac{\partial N_{t+k|t}}{\partial \tilde{W}_t} + \lambda_{t+k|t}\left(N_{t+k|t} + \tilde{W}_t\frac{\partial N_{t+k|t}}{\partial \tilde{W}_t}\right)\right] = 0 \,. \text{ (1-1-91)}
$$

Notice that $-N_{t+k|t}^{\varphi} \equiv \dfrac{\partial \tilde{U}_{t+k}}{\partial N_{t+k|t}}$.

Partial derivative of Eq.(1-1-89) is given by:

$$
\begin{aligned}
\frac{\partial N_{t+k|t}}{\partial \tilde{W}_t} &= -\varepsilon_W \left(\frac{\tilde{W}_t}{W_{t+k}}\right)^{-\varepsilon_W - 1}\frac{1}{W_{t+k}} N_{t+k} \\
&= -\varepsilon_W \left(\frac{\tilde{W}_t}{W_{t+k}}\right)^{-\varepsilon_W} N_{t+k}\frac{W_{t+k}}{\tilde{W}_{H,t}}\frac{1}{W_{t+k}} \,, \text{ (1-1-92)} \\
&= -\varepsilon_W N_{t+k|t}\frac{1}{\tilde{W}_t}
\end{aligned}
$$

where Eq.(1-1-89) is used to eliminate $\left(\dfrac{\tilde{W}_t}{W_{t+k}}\right)^{-\varepsilon_W} N_{t+k}$ in the second line.

Plugging Eq.(1-1-92) into Eq.(1-1-91) yields:

$$
\begin{aligned}
\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left\{\begin{matrix}\varepsilon_W N_{t+k|t}^{1+\varphi} Z_{t+k}\dfrac{1}{\tilde{W}_{H,t}} \\ +\lambda_{t+k|t}\left[N_{t+k|t} + (-\varepsilon_W)N_{t+k|t}\right]\end{matrix}\right\} &= \sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left[\begin{matrix}\varepsilon_W N_{t+k|t}^{1+\varphi} Z_{t+k}\dfrac{1}{\tilde{W}_{H,t}} \\ +\lambda_{t+k|t}(1-\varepsilon_W)N_{t+k|t}\end{matrix}\right] \\
&= 0
\end{aligned}
$$

Multiplying −1 on both sides of the previous expression yields:

$$
\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left[-\varepsilon_W N_{t+k|t}^{1+\varphi} Z_{t+k}\frac{1}{\tilde{W}_{H,t}} + \lambda_{t+k|t}(\varepsilon_W - 1)N_{t+k|t}\right] = 0
$$

Plugging $\lambda_{t+k|t} = \dfrac{1}{P_{t+k}C_{t+k}}$ into the previous expression yields:

$$\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left[-\varepsilon_W N_{t+k|t}^{1+\varphi}\frac{1}{\tilde{W}_t}+\frac{1}{P_{t+k}}\frac{N_{t+k|t}}{C_{t+k}}(\varepsilon_W-1)\right]=0\,.$$

Multiplying both sides on the previous expression by $\dfrac{\tilde{W}_t}{\varepsilon_W-1}$ yields:

$$\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+k|t}^{1+\varphi}+\frac{\tilde{W}_t}{P_{t+k}}\frac{N_{t+k|t}}{C_{t+k}}\right)=0\,. \quad (1\text{-}1\text{-}93)$$

The LHS of Eq.(1-1-93) can be rewritten as:

$$\begin{aligned}
&\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+k|t}^{1+\varphi}+\frac{\tilde{W}_t}{P_{t+k}}\frac{N_{t+k|t}}{C_{t+k}}\right)\\
&=-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t|t}^{1+\varphi}+\frac{\tilde{W}_t}{P_t}\frac{N_{t|t}}{C_t}+\beta\theta_W\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+1|t}^{1+\varphi}+\frac{\tilde{W}_t}{P_{t+1}}\frac{N_{t+1|t}}{C_{t+1}}\right)\\
&+(\beta\theta_W)^2\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+2|t}^{1+\varphi}+\frac{\tilde{W}_t}{P_{t+2}}\frac{N_{t+2|t}}{C_{t+2}}\right)+\cdots\\
&=-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t|t}^{\varphi}C_t\frac{N_{t|t}}{C_t}+\frac{\tilde{W}_t}{P_t}\frac{N_{H,t|t}}{C_t}+\beta\theta_W\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+1|t}^{\varphi}C_{t+1}\frac{N_{t+1|t}}{C_{t+1}}+\frac{\tilde{W}_t}{P_{t+1}}\frac{N_{t+1|t}}{C_{t+1}}\right)\\
&+(\beta\theta_W)^2\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+2|t}^{1+\varphi}C_{t+2}\frac{N_{t+2|t}}{C_{t+2}}+\frac{\tilde{W}_t}{P_{t+2}}\frac{N_{t+2|t}}{C_{t+2}}\right)+\cdots\\
&=-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t|t}^{\varphi}C_t\frac{N_{t|t}}{C_t}+\frac{\tilde{W}_t}{P_t}\frac{N_{t|t}}{C_t}+\beta\theta_W\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+1|t}^{\varphi}C_{t+1}\frac{N_{t+1|t}}{C_{t+1}}+\frac{\tilde{W}_t}{P_{t+1}}\frac{N_{t+1|t}}{C_{t+1}}\right)\\
&+(\beta\theta_W)^2\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+2|t}^{1+\varphi}C_{t+2}\frac{N_{t+2|t}}{C_{t+2}}+\frac{\tilde{W}_t}{P_{t+2}}\frac{N_{t+2|t}}{C_{t+2}}\right)+\cdots\\
&=-\frac{\varepsilon_W}{\varepsilon_W-1}MRS_{t|t}\frac{N_{t|t}}{C_t}+\frac{\tilde{W}_t}{P_t}\frac{N_{t|t}}{C_t}+\beta\theta_W\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}MRS_{t+1|t}\frac{N_{t+1|t}}{C_{t+1}}+\frac{\tilde{W}_t}{P_{t+1}}\frac{N_{t+1|t}}{C_{t+1}}\right).\\
&+(\beta\theta_W)^2\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}MRS_{t+2|t}\frac{N_{t+2|t}}{C_{t+2}}+\frac{\tilde{W}_t}{P_{t+2}}\frac{N_{t+2|t}}{C_{t+2}}\right)+\cdots=0
\end{aligned}$$

Note that:

$$MRS_{t+k|t}\equiv-\frac{\partial U_{t+k|t}/\partial N_{t+k|t}}{\partial U_{t+k}/\partial C_{t+k}}=\frac{N_{t+k|t}^{\varphi}}{C_{t+k}^{-1}}\,. \quad (1\text{-}1\text{-}93)'$$

Further, the last equality can be rewritten as:

$$\frac{\varepsilon_W}{\varepsilon_W-1}\begin{bmatrix} MRS_{t|t}\frac{N_{t|t}}{C_t}+\beta\theta_W \mathrm{E}_t\left(MRS_{H,t+1|t}\frac{N_{t+1|t}}{C_{t+1}}\right) \\ +(\beta\theta_W)^2\mathrm{E}_t\left(MRS_{t+2|t}\frac{N_{t+2|t}}{C_{t+2}}\right)\cdots \end{bmatrix}=\tilde{W}_t\begin{bmatrix} \frac{N_{t|t}}{P_tC_t}+\beta\theta_W \mathrm{E}_t\left(\frac{N_{t+1|t}}{P_{t+1}C_{t+1}}\right) \\ +(\beta\theta_W)^2\mathrm{E}_t\left(\frac{N_{t+2|t}}{P_{t+2}C_{t+2}}\right)\cdots \end{bmatrix},$$

which can be a compact form as follows:

$$\frac{\varepsilon_W}{\varepsilon_W-1}\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left(MRS_{t+k|t}N_{t+k|t}C_{t+k}^{-1}\right)=\tilde{W}_t\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left[N_{t+k|t}\left(P_{t+k}C_{t+k}\right)^{-1}\right]. \quad (1\text{-}1\text{-}94)$$

Eq.(1-1-94) can be rewritten as:

$$\tilde{W}_t=\frac{\varepsilon_W/(\varepsilon_W-1)\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left(MRS_{t+k|t}N_{t+k|t}C_{t+k}^{-1}\right)}{\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left[N_{t+k|t}\left(P_{t+k}C_{t+k}\right)^{-1}\right]}.$$

(1-1-95)

Given the assumed wage structure, the evolution of the aggregate wage index is given by:

$$W_t=\left[\theta_W W_{t-1}^{1-\varepsilon_W}+(1-\theta_W)\tilde{W}_t^{1-\varepsilon_W}\right]^{\frac{1}{1-\varepsilon_W}}. \quad (1\text{-}1\text{-}97)$$

2 Nonstochastic Steady State

We focus on equilibria where the state variables follow paths that are close to a deterministic stationary equilibrium, in which $\Pi_{H,t}=\Pi_{F,t}=\Pi_t=1$, $S=1$ and $\beta=R^{-1}=\left(R^*\right)^{-1}$. Because this steady state is nonstochastic, the productivity has unit values; i.e., $A=1$.

2.2 Steady State Relative Price and Market Clearing

In the steady state, we assume $P_H=P_F$ and then $S=1$ is applied. Because of $S=1$, $P_H^*=P_F^*$ is applied. The CPI implies that:

$$\begin{aligned} P_t &\equiv \left[\upsilon P_{H,t}^{1-\eta} + (1-\upsilon) P_{H,t}^{1-\eta}\right]^{\frac{1}{1-\eta}} \\ &= \left(P_{H,t}^{1-\eta}\right)^{\frac{1}{1-\eta}} \\ &= P_{H,t} \end{aligned}.$$

That is, we have:

$$\begin{aligned} P &= P_H = P_F \\ P^* &= P_H^* = P_F^* \end{aligned}. \quad (4\text{-}6\text{-}3)$$

In addition, due to $P_{H,t} = P_{H,t}^*$, we have:

$$\begin{aligned} P_t &\equiv \left[(1-\upsilon)\left(P_{H,t}^*\right)^{1-\eta} + \upsilon\left(P_{H,t}^*\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \\ &= P_{H,t}^* \end{aligned},$$

which implies that:

$P = P_H^* = P_F^*$. (4-14-17)

Plugging Eq.(4-6-3) into Eq.(4-14-17) yields:

$P = P^*$,

which implies that the PPP is applicable in the steady state. Plugging the previous expression into the definition of the CPI disparity yields:

$Q = 1$. (2-7)

Plugging Eq.(2-7) into the international risk sharing condition $C = QC^*\vartheta$, we have:

$C = C^*$ (2-8),

where we impose $\vartheta = 1$.

Market clearing conditions in the steady state are given by:

$$\begin{aligned} Y &= C \\ Y^* &= C^* \end{aligned} \quad (2\text{-}9)$$

2.4 Steady State Wedge between Marginal Utility of Consumption and Labor

The FONC for households to choose optimal wage Eq.(1-1-93) implies that:

$$-\frac{\varepsilon_W}{\varepsilon_W-1}N^{1+\varphi}+\frac{W}{P}\frac{N}{C}+\beta\theta_W\left[-\frac{\varepsilon_W}{\varepsilon_W-1}N^{1+\varphi}+\frac{W}{P}\frac{N}{C}\right]$$
$$+\left(\beta\theta_W\right)^2\left[-\frac{\varepsilon_W}{\varepsilon_W-1}N^{1+\varphi}+\frac{W}{P}\frac{N}{C}\right]+\cdots=0,$$

which can be rewritten as:

$$\left[1+\beta\theta_W\left(\beta\theta_W\right)^2+\cdots\right]\frac{W}{P}\frac{N}{C}=\left[1+\beta\theta_W\left(\beta\theta_W\right)^2+\cdots\right]\frac{\varepsilon_W}{\varepsilon_W-1}N^{1+\varphi}.$$

Then, we have:

$$\frac{W}{P}=\frac{\varepsilon_W}{\varepsilon_W-1}N^{\varphi}C. \quad (2\text{-}27)$$

Eq. (1-1-83)implies as follows:

$$MC=\frac{1}{1-\tau}\frac{W}{P}. \quad (2\text{-}27)'$$

Combining Eqs.(2-27) and (2-27)' yields:

$$MC\left(1-\tau\right)=\frac{\varepsilon_W}{\varepsilon_W-1}N^{\varphi}C. \quad (2\text{-}27)''$$

The FONC for the firms Eq.(1-1-82) implies that:

$$1=\frac{\varepsilon_p}{\varepsilon_p-1}MC,$$

which can be rewritten as:

$$MC=\frac{\varepsilon_p-1}{\varepsilon_p}. \quad (2\text{-}28)$$

Because of Eqs(2-30), (2-32) and (2-32)', Eq.(2-27)'' can be rewritten as:

$$\frac{\varepsilon_W}{\varepsilon_W-1}N^{\varphi}C=\frac{\left(\varepsilon_p-1\right)\left(1-\tau\right)}{\varepsilon_p}. \quad (2\text{-}33)$$

Note that $U_C=C^{-1}$ and $-U_N=N^{\varphi}$. Plugging these conditions into Eq.(2-33) yields:

$$-\frac{U_N}{U_C}=\frac{1-\tau}{MM^W}. \quad (2\text{-}34)$$

with $M^p\equiv\frac{\varepsilon_p}{\varepsilon_p-1}$ and $M^W\equiv\frac{\varepsilon_W}{\varepsilon_W-1}$.

Plugging $-\frac{U_N}{U_C}\equiv 1-\Phi$ into the previous expression, we have:

$$1-\Phi=\frac{1-\tau}{\mathcal{M}^{p}\mathcal{M}^{w}} \quad \text{or} \quad \Phi=1-\frac{1-\tau}{\mathcal{M}^{p}\mathcal{M}^{w}}, \text{ (2-35)}$$

which shows the steady state wedge between marginal utility of consumption and its labor.

3 Log-linearization of the Model

3.1 Distorted Steady State Model

3.1.1 Relationship between Real Exchange Rate and Terms of Trade

Note that:

$$\begin{aligned} q_t &= e_t + p_t^* - p_t \\ &= e_t + p_{F,t}^* - p_t \\ &= p_{F,t} - (1-\upsilon)p_{H,t} - \alpha p_{F,t} \text{ . (3-1-4)} \\ &= (1-\upsilon)(p_{F,t} - p_{H,t}) \\ &= (1-\upsilon)s_t \end{aligned}$$

3.1.2 Market Clearing Condition

By Using $X_{H,t} \equiv P_{H,t}/P_t$, Eq.(1-1-76) can be rewritten as:

$$Y_t = (1-\upsilon)X_{H,t}^{-\eta}C_t + \upsilon S_t^{\eta} Z_{1,t}^*,$$

Where we use Eq.(1-1-20).

Total derivative of the previous expression is given by:

$$dY_t = (1-\upsilon)(-\eta)CdX_{H,t} + (1-\upsilon)dC_t + \upsilon\eta CdS_t + \upsilon dZ_{1,t}^* .$$

Dividing both sides by *Y* yields:

$$\frac{dY_t}{Y} = -\eta(1-\upsilon)\frac{C}{Y}dX_{H,t} + (1-\upsilon)\frac{C}{Y}\frac{dC_t}{C} + \upsilon\eta\frac{C}{Y}dS_t + \upsilon\frac{C}{Y}\frac{Z_1^*}{C^*}\frac{dZ_{1,t}^*}{Z_1^*},$$

which can be rewritten as:

$$y_t = -\eta(1-\upsilon)x_t + \eta\upsilon s_t + (1-\upsilon)c_t + \upsilon z_{1,t}^* \quad \text{(3-1-6)}$$

where $C/Y = 1$. Note that:

$$\begin{aligned} x_{H,t} &= p_{H,t} - p_t \\ &= p_{H,t} - (1-\upsilon) p_{H,t} - \upsilon p_{F,t} \\ &= -\upsilon\left(p_{F,t} - p_{H,t}\right) \\ &= -\upsilon s_t \end{aligned} \quad \text{(3-1-7)}$$

Plugging the last line in Eq.(3-1-7) into Eq(3-1-6) yields:

$$\begin{aligned} y_t &= -\eta(1-\upsilon)(-\upsilon s_t) + \eta\upsilon s_t + (1-\upsilon)c_t + \upsilon z_{1,t}^* \\ &= \eta\upsilon(2-\upsilon)s_t + (1-\upsilon)c_t + \upsilon z_{1,t}^* \end{aligned} \quad \text{(3-1-8)' [(16) in the text]}$$

Total derivative of Eq.(3-1-77) is given by:

$$dY_t^* = dC_t^* .$$

Dividing both sides by Y yields:

$$\frac{dY_t^*}{Y^*} = \frac{dC_t^*}{C^*},$$

which can be rewritten as:

$$y_t^* = c_t^* . \quad \text{(3-1-9)}$$

3.1.3 International Risk Sharing Condition

Total derivative of Eq.(1-1-21) is given by:

$$\begin{aligned} dC_t &= dQ_t + \frac{1}{Z_2^*} dZ_t - Z\left(Z_2^*\right)^{-2} dZ_{2,t}^* \\ &= \frac{Z}{Z_2^*} dQ_t + \frac{Z}{Z_2^*}\frac{dZ_t}{Z} - \frac{Z}{Z_2^*}\frac{dZ_{2,t}^*}{Z_2^*} \end{aligned}$$

Dividing both sides by C yields:

$$\begin{aligned} \frac{dC_t}{C} &= \frac{Z_2^*}{Z}\left(\frac{Z}{Z_2^*} dQ_t + \frac{Z}{Z_2^*}\frac{dZ_t}{Z} - \frac{Z}{Z_2^*}\frac{dZ_{2,t}^*}{Z_2^*}\right) \\ &= dQ_t + \frac{dZ_t}{Z} - \frac{dZ_{2,t}^*}{Z_2^*} \end{aligned},$$

which can be rewritten as:

$$c_t = q_t + z_t - z_{2,t}^* \quad \text{(3-1-10)}$$

Plugging the last line of Eq.(3-1-4) into Eq.(3-1-10) yields:

$$c_t = (1-\upsilon)s_t + z_t - z_{2,t}^* . \quad \text{(3-1-11) [(17) in the text]}$$

Plugging Eq.(3-1-11) into Eq.(3-1-8)' yields:

$$
\begin{aligned}
y_t &= \eta\upsilon(2-\upsilon)s_t + (1-\upsilon)\left[(1-\upsilon)s_t + z_t - z_{2,t}^*\right] + \upsilon z_{1,t}^* \\
&= \left[\eta\upsilon(2-\upsilon) + (1-\upsilon)^2\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \\
&= \left[\eta\upsilon(2-\upsilon) + (1-\upsilon)^2\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \\
&= \left(2\eta\upsilon - \eta\upsilon^2 + 1 - 2\upsilon + \upsilon^2\right)s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \quad . \text{(3-1-12)} \\
&= \left[2\upsilon(\eta-1) - \upsilon^2(\eta-1) + 1\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \\
&= \left[(\eta-1)\left(2\upsilon - \upsilon^2\right) + 1\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \\
&= \left[(\eta-1)\upsilon(2-\upsilon) + 1\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^*
\end{aligned}
$$

Note that $\sigma_\varsigma \equiv \left[(\eta-1)\upsilon(2-\upsilon)+1\right]^{-1}$

When $\eta = 1$, Eq.(3-1-12) boils down to:

$$y_t = (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^*$$

3.1.4 Euler Equation

Log-linearizing Eq.(1-1-7) yields:

$$c_t = \mathrm{E}_t\left(c_{t+1}\right) - r_t + \mathrm{E}_t\left(\pi_{t+1}\right) + (1-\rho_z)z_t + \delta \text{, (3-1-16) [(18) in the text]}$$

with $\delta \equiv \beta^{-1} - 1 = r$ being time discount rate where we use the fact that

$$\mathrm{E}_t\left(z_{t+1}\right) = \rho_z z_t \, .$$

3.1.5 Price Index and the Definition of the TOT

The definition of the TOT is given by $S_t \equiv P_{F,t}/P_{H,t}$, which can be log-linearized as:

$$
\begin{aligned}
s_t &= p_{F,t} - p_{H,t} \\
&= e_t + p_{F,t}^* - p_{H,t} \text{ . (3-1-17)' [(19) in the text]} \\
&= e_t - p_{H,t}
\end{aligned}
$$

Log-linearizing Eq.(1-1-78) yields:

$$
\begin{aligned}
p_t &= (1-\upsilon)p_{H,t} + \upsilon p_{F,t} \\
&= p_{H,t} + \upsilon\left(p_{F,t} - p_{H,t}\right) \text{. (3-1-17) [(25) in the text]} \\
&= p_{H,t} + \upsilon s_t
\end{aligned}
$$

Combining the previous expression and its lagged expression yields:

$\pi_t = (1-\upsilon)\pi_{H,t} + \upsilon\pi_{F,t}$,

Where:

$\pi_{H,t} \equiv p_{H,t} - p_{H,t-1}$. (3-1-17)'' [(23) in the text]

because of inflation targeting in the ROW, $p^*_{F,t} = 0$ for all *t*.

Combining the previous expression and its lagged expression yields:

$s_t - s_{t-1} = \pi_{F,t} - \pi_{H,t}$ (3-1-18)

Plugging Eq(3-1-18) into eq.(3-1-17) yields:

$$\begin{aligned}\pi_t &= (1-\upsilon)\pi_{H,t} + \upsilon\left(\pi_{H,t} + s_t - s_{t-1}\right) \\ &= \pi_{H,t} + \upsilon s_t - \upsilon s_{t-1}\end{aligned}.$$

The definition of the CPI $\Pi_t \equiv \frac{P_t}{P_{t-1}}$ can be log-linearized as:

$\pi_t \equiv p_t - p_{t-1}$. (3-1-19) [(24) in the text]

3.1.7 NKPC

The NKPC is given by:

$\pi_{H,t} = \beta \mathrm{E}_t\left(\pi_{H,t+1}\right) + \kappa mc_t$, (3-1-23) [(21) in the text]

with $\kappa_p \equiv \frac{\left(1-\theta_p\right)\left(1-\beta\theta_p\right)}{\theta_p}$.

Total derivative of Eq.(1-1-83) yields:

$$\begin{aligned}dMC_t &= \left(\frac{W}{P}\right)(-1)(1-\tau)^{-2}(-1)d\tau_t + \frac{1}{1-\tau}d\left(\frac{W_t}{P_{H,t}}\right) + \frac{1}{1-\tau}\left(\frac{W}{P}\right)(-1)dA_t \\ &= \frac{1}{1-\tau}\left(\frac{W}{P}\right)\frac{\tau}{1-\tau}\frac{d\tau_t}{\tau} + \frac{1}{1-\tau}d\left(\frac{W_t}{P_{H,t}}\right) - \frac{1}{1-\tau}\left(\frac{W}{P}\right)dA_t\end{aligned}.$$

Dividing both sides of the previous expression by $MC = \frac{1}{1-\tau}\frac{W}{P}$ yields:

$$\frac{dMC_t}{MC} = \frac{1}{1-\tau}d\tau_t + \frac{d\left(W_{H,t}/P_{H,t}\right)}{(W/P)} - dA_{H,t}.$$

As mentioned, $d\tau_t = \tau_t - \tau$. Thus, we have:

$$
\begin{aligned}
mc_t &= \frac{1}{1-\tau}\tau_t + w_t^r - a_t - \frac{1}{1-\tau}\tau \\
&= \left(w_t - p_{H,t}\right) - a_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&= w_t - p_t + p_t - p_{H,t} - a_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&= \omega_t + (1-\upsilon)p_{H,t} + \upsilon p_{F,t} - p_{H,t} - a_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&= \omega_t + \upsilon\left(p_{F,t} - p_{H,t}\right) - a_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&= \omega_t + \upsilon s_t - a_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau
\end{aligned}
, \quad (3\text{-}1\text{-}24)
$$

with $w_{H,t}^r \equiv \frac{d\left(W_{H,t}/P_{H,t}\right)}{\left(W/P\right)}$ being the (log) real wage.

Eq.(1-1-80) can be rewritten as:

$$
N_t = \left(\frac{Y_t \Delta_{p,t}}{A_t}\right)^{\frac{1}{1-\alpha}}.
$$

Total differential of the previous expression yields:

$$
\begin{aligned}
dN_t &= \frac{1}{1-\alpha}Y^{\frac{1}{1-\alpha}-1}dY_t + \frac{1}{1-\alpha}Y^{\frac{1}{1-\alpha}-1}Y(-1)dA_t + \frac{1}{1-\alpha}Y^{\frac{1}{1-\alpha}-1}Yd\Delta_{p,t} \\
&= \frac{1}{1-\alpha}Y^{\frac{1}{1-\alpha}}\frac{dY_t}{Y} - \frac{1}{1-\alpha}Y^{\frac{1}{1-\alpha}}dA_t + \frac{1}{1-\alpha}Y^{\frac{1}{1-\alpha}}d\Delta_{p,t}
\end{aligned}
,
$$

where $\ln\Delta_{p,t}$ is $o\left(\|\xi\|^2\right)$ and is omitted.

Dividing both sides on the previous expression by N yields:

Second-order approximation of Eq.(1-1-80) is given by:

$$
\begin{aligned}
\frac{dN_t}{N} &= Y^{-1}\left(Y\frac{dY_t}{Y} - YdA_t + Yd\Delta_{p,t}\right) \\
&= \frac{dY_t}{Y} - dA_t + d\Delta_{p,t}
\end{aligned}
,
$$

which can be rewritten as:

$n_t = y_t - a_t$. (3-1-26) [(20) in the text]

where $\ln\Delta_{p,t}$ is $o\left(\|\xi\|^2\right)$ and is omitted.

3.1.8 Wage Inflation Dynamics

Total derivative of Eq.(1-1-95) is given by:

$$
\begin{aligned}
d\tilde{W}_t = \mathcal{M}^w & \begin{Bmatrix} (-1)C^{-2}NMRSdC_t + C^{-1}MRSd\mathcal{N}_t + C^{-1}Nd\mathcal{MRS}_t \\ +\beta\theta_w\left[(-1)C^{-2}NMRSdC_{t+1} + C^{-1}MRSd\mathcal{N}_{t+1|t} + C^{-1}Nd\mathcal{MRS}_{t+1|t}\right] \\ +(\beta\theta_w)^2\left[(-1)C^{-2}NMRSdC_{t+1} + C^{-1}MRSd\mathcal{N}_{t+1|t} + C^{-1}Nd\mathcal{MRS}_{t+1|t}\right] \\ +\cdots \end{Bmatrix} \\
& \times\left\{C^{-1}NP^{-1}\left[1+\beta\theta_w+(\beta\theta_w)^2+\cdots\right]\right\}^{-1} \\
& +(-1)\left\{C^{-1}NP^{-1}\left[1+\beta\theta_w+(\beta\theta_w)^2+\cdots\right]\right\}^{-2} \\
& \times\begin{Bmatrix} (-1)C^{-2}NP^{-1}dC_t + C^{-1}P^{-1}d\mathcal{N}_t + (-1)C^{-1}NP^{-2}dP_t \\ +\beta\theta_w\left[(-1)C^{-2}NP^{-1}dC_{t+1} + C^{-1}P^{-1}d\mathcal{N}_{t+1|t} + (-1)C^{-1}NP^{-2}dP_{t+1}\right] \\ +(\beta\theta_w)^2\left[(-1)C^{-2}NP^{-1}dC_{t+2} + C^{-1}P^{-1}d\mathcal{N}_{t+2|t} + (-1)C^{-1}NP^{-2}dP_{t+2}\right] \\ +\cdots \end{Bmatrix} \\
& \times\mathcal{M}^w C^{-1}NMRS\left[1+\beta\theta_w+(\beta\theta_w)^2+\cdots\right]
\end{aligned}
.
$$

Then:

$$d\tilde{W}_t = M^w MRSC^{-1}N \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} + \frac{dMRS_t}{MRS} \\ +\beta\theta_w \left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} + \frac{dMRS_{t+1|t}}{MRS} \right) \\ (\beta\theta_w)^2 \left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} + \frac{dMRS_{t+2|t}}{MRS} \right) + \cdots \end{bmatrix} CN^{-1}P \left(\frac{1}{1-\beta\theta_w} \right)^{-1}$$

$$-M^w MRSCN^{-1}P^2C^{-1}NP^{-1} \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} - \frac{dP_t}{P} \\ +\beta\theta_w \left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} - \frac{dP_{t+1}}{P} \right) \\ (\beta\theta_w)^2 \left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} - \frac{dP_{t+2}}{P} \right) + \cdots \end{bmatrix} \left(\frac{1}{1-\beta\theta_w} \right)^{-1}$$

$$= M^w PMRS \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} + \frac{dMRS_t}{MRS} \\ +\beta\theta_w \left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} + \frac{dMRS_{t+1|t}}{MRS} \right) \\ (\beta\theta_w)^2 \left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} + \frac{dMRS_{t+2|t}}{MRS} \right) + \cdots \end{bmatrix} \left(\frac{1}{1-\beta\theta_w} \right)^{-1}$$

$$-M^w PMRS \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} - \frac{dP_t}{P} \\ +\beta\theta_w \left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} - \frac{dP_{t+1}}{P} \right) \\ (\beta\theta_w)^2 \left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} - \frac{dP_{t+2}}{P} \right) + \cdots \end{bmatrix} \left(\frac{1}{1-\beta\theta_w} \right)^{-1} .$$

Note that $\tilde{W} = PM^w MRS$. Thus:

$$\frac{d\tilde{W}_t}{\tilde{W}} = \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} + \frac{dMRS_t}{MRS} \\ +\beta\theta_w\left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} + \frac{dMRS_{t+1|t}}{MRS}\right) \\ \left(\beta\theta_w\right)^2\left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} + \frac{dMRS_{t+2|t}}{MRS}\right) + \cdots \end{bmatrix}\left(\frac{1}{1-\beta\theta_w}\right)^{-1}$$

$$-\begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} - \frac{dP_t}{P} \\ +\beta\theta_w\left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} - \frac{dP_{t+1}}{P}\right) \\ +\left(\beta\theta_w\right)^2\left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} - \frac{dP_{t+2}}{P}\right) + \cdots \end{bmatrix}\left(\frac{1}{1-\beta\theta_w}\right)^{-1},$$

$$= \left(1-\beta\theta_w\right)\begin{bmatrix} \frac{dMRS_t}{MRS} + \beta\theta_w\frac{dMRS_{t+1|t}}{MRS} + \left(\beta\theta_w\right)^2\frac{dMRS_{t+2|t}}{MRS} \\ + \cdots + \frac{dC_t}{C} + \beta\theta_w\frac{dC_{t+1}}{C} + \left(\beta\theta_w\right)^2\frac{dC_{t+2}}{C} + \cdots \end{bmatrix}$$

which can be rewritten as:

$\tilde{w}_t = \left(1-\beta\theta_w\right)\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k \mathrm{E}_t\left(mrs_{t+k|t} + p_{t+k}\right)$. (3-1-27)

Total derivative of Eq.(1-1-93)' is given by:

$$\begin{aligned} dMRS_{t+k|t} &= C\varphi N^{\varphi-1}dN_{t+k|t} + N^{\varphi}dC_{t+k} \\ &= CN^{\varphi}\varphi\frac{dN_{t+k|t}}{N} + CN^{\varphi}\frac{dC_{t+k}}{C}. \end{aligned}$$

Note that $MRS = CN^{\varphi}$. Dividing both sides of the previous expression by MRS_H yields:

$$\frac{dMRS_{t+k|t}}{MRS} = \varphi\frac{dN_{t+k|t}}{N} + \frac{dC_{t+k}}{C},$$

which can be rewritten as:

$mrs_{t+k|t} = \varphi n_{t+k|t} + c_{t+k}$. (3-1-28)

Let define $MRS_{H,t+k} \equiv \frac{N_{H,t+k}^{\varphi}}{C_{t+k}}$. Total derivative of this definition is given by:

$$dMRS_{t+k} = C\varphi N^{\varphi-1}dN_{t+k} + N^{\varphi}dC_{t+k}$$
$$= CN^{\varphi}\varphi\frac{dN_{t+k}}{N} + CN^{\varphi}\frac{dC_{t+k}}{C},$$

Dividing both sides of the previous expression yields:

$$\frac{dMRS_{t+k}}{MRS} = \varphi\frac{dN_{t+k}}{N} + \frac{dC_{t+k}}{C},$$

because of $MRS_H = CN^{\varphi}$.

That expression can be rewritten as:

$mrs_{t+k} = \varphi n_{t+k} + c_{t+k}$. (3-1-29)

Subtracting Eq.(3-1-29) from Eq.(3-1-28) yields:

$mrs_{t+k|t} = mrs_{t+k} + \varphi\left(n_{t+k|t} - n_{t+k}\right)$. (3-1-30)

Total derivative of Eq.(1-1-89) is given by:

$$dN_{t+k|t} = -\varepsilon_W W^{-\varepsilon_W-1}W^{\varepsilon_W}Nd\tilde{W}_t + \varepsilon_W W^{\varepsilon_W-1}W^{-\varepsilon_W}NdW_{t+k} + dN_{t+k}$$
$$= -\varepsilon_W N\frac{d\tilde{W}_t}{\tilde{W}} + \varepsilon_W N\frac{dW_{t+k}}{W} + dN_{t+k}.$$

Dividing both sides of previous expression by $N = N$ yields:

$$\frac{dN_{t+k|t}}{N} = -\varepsilon_W\frac{d\tilde{W}_t}{\tilde{W}} + \varepsilon_W\frac{dW_{t+k}}{W} + \frac{dN_{t+k}}{N},$$

Which can be rewritten as:

$n_{t+k|t} = -\varepsilon_W\left(\tilde{w}_t - w_{t+k}\right) + n_{t+k}$. (3-1-31)

Plugging Eq.(3-1-31) into Eq.(3-1-30) yields:

$mrs_{t+k|t} = mrs_{t+k} - \varepsilon_w\varphi\left(\tilde{w}_t - w_{t+k}\right)$. (3-1-32)

Plugging Eq.(3-1-32) into Eq.(3-1-27) yields:

$$
\begin{aligned}
\tilde{w}_{H,t} &= (1-\beta\theta_w)\sum_{k=0}^{\infty}(\beta\theta_w)^k \mathrm{E}_t\left[mrs_{t+k}-\varepsilon_w\varphi(\tilde{w}_t-w_{t+k})+p_{t+k}\right] \\
&= (1-\beta\theta_w)\begin{Bmatrix} mrs_t-\varepsilon_w\varphi(\tilde{w}_t-w_t)+p_t+\beta\theta_w\left[mrs_{t+1}-\varepsilon_w\varphi(\tilde{w}_t-w_{t+1})+p_{t+1}\right] \\ +(\beta\theta_w)^2\left[mrs_{t+2}-\varepsilon_w\varphi(\tilde{w}_t-w_{t+2})+p_{t+2}\right]+\cdots \end{Bmatrix} \\
&= (1-\beta\theta_w)\begin{Bmatrix} mrs_t+\varepsilon_w\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_w)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \\ -\varepsilon_w\varphi\tilde{w}_t-\varepsilon_w\varphi\beta\theta_w\tilde{w}_t-(\beta\theta_w)^2\varepsilon_w\varphi\tilde{w}_t \end{Bmatrix} \\
&= (1-\beta\theta_w)\begin{Bmatrix} mrs_t+\varepsilon_w\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_w)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \\ -\varepsilon_w\varphi\tilde{w}_t\left[1+\beta\theta_w+(\beta\theta_w)^2+\cdots\right] \end{Bmatrix} \\
&= (1-\beta\theta_w)\begin{bmatrix} mrs_t+\varepsilon_w\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_w)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \\ -\varepsilon_w\varphi\tilde{w}_t\left(\dfrac{1}{1-\beta\theta_w}\right) \end{bmatrix}, \\
&= (1-\beta\theta_w)\begin{bmatrix} mrs_t+\varepsilon_w\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_w)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \end{bmatrix}-\varepsilon_w\varphi\tilde{w}_t
\end{aligned}
$$

which can be rewritten as:

$$
(1+\varepsilon_w\varphi)\tilde{w}_t=(1-\beta\theta_w)\begin{bmatrix} mrs_t+\varepsilon_w\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_w)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \end{bmatrix}.
$$

This expression can be the compact form as follows:

$$
\tilde{w}_t=\frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\sum_{k=0}^{\infty}(\beta\theta_w)^k \mathrm{E}_t\left(mrs_{t+k}+\varepsilon_w\varphi w_{t+k}+p_{t+k}\right). \quad (3\text{-}1\text{-}33)
$$

Let define $\mu_t^w \equiv \omega_t - mrs_t$ which is the (log) average wage markup in the SOE. This expression can be rewritten as:

$\mu_t^w = \omega_t - \varphi n_{t+k} - c_{t+k}$,

By using Eq.(3-1-29).

Plugging the definition of the (log) average wage markup into Eq.(3-1-33) yields:

$$\tilde{w}_t = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\sum_{k=0}^{\infty}(\beta\theta_w)^k \mathrm{E}_t\left[(1+\varepsilon_w\varphi)w_{t+k} - \mu_{t+k}^w\right]. \text{ (3-1-35)}$$

Eq.(3-1-35) can be rewritten as:

$$\begin{aligned}\tilde{w}_t &= \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\mathrm{E}_t\begin{Bmatrix}(1+\varepsilon_w\varphi)w_t - \mu_t^w + \beta\theta_w\left[(1+\varepsilon_w\varphi)w_{t+1} - \mu_{t+1}^w\right] \\ +(\beta\theta_w)^2\left[(1+\varepsilon_w\varphi)w_{t+2} - \mu_{t+2}^w\right] + \cdots\end{Bmatrix} \\ &= \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\left[(1+\varepsilon_w\varphi)w_t - \mu_t^w\right] + \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\mathrm{E}_t\begin{Bmatrix}\beta\theta_w\left[(1+\varepsilon_w\varphi)w_{t+1} - \mu_{t+1}^w\right] \\ +(\beta\theta_w)^2\left[(1+\varepsilon_w\varphi)w_{t+2} - \mu_{t+2}^w\right] \\ +\cdots\end{Bmatrix}\end{aligned}. \text{ (3-1-36)}$$

Forwarding Eq.(3-1-36) one period yield:

$$\mathrm{E}_t(\tilde{w}_{t+1}) = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\mathrm{E}_t\begin{Bmatrix}(1+\varepsilon_w\varphi)w_{t+1} - \mu_{t+1}^w + \beta\theta_w\left[(1+\varepsilon_w\varphi)w_{t+2} - \mu_{t+2}^w\right] \\ +(\beta\theta_w)^2\left[(1+\varepsilon_w\varphi)w_{t+3} - \mu_{t+3}^w\right] + \cdots\end{Bmatrix}$$

Multiplying $\beta\theta_W$ both sides of the previous expression yields:

$$\beta\theta_w\mathrm{E}_t(\tilde{w}_{t+1}) = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\mathrm{E}_t\begin{Bmatrix}\beta\theta_w\left[(1+\varepsilon_w\varphi)w_{t+1} - \mu_{t+1}^w\right] + (\beta\theta_w)^2\left[(1+\varepsilon_w\varphi)w_{t+2} - \mu_{t+2}^w\right] \\ +(\beta\theta_w)^3\left[(1+\varepsilon_w\varphi)w_{t+3} - \mu_{t+3}^w\right] + \cdots\end{Bmatrix}$$

Plugging the previous expression into Eq.(3-1-36) yields:

$$\tilde{w}_t = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\left[(1+\varepsilon_w\varphi)w_t - \mu_t^w\right] + \beta\theta_w\mathrm{E}_t(\tilde{w}_{t+1}). \text{ (3-1-37)}$$

Total derivative of Eq.(1-1-97) is given by:

$$\begin{aligned}dW_t &= \frac{1}{1-\varepsilon_w}\left(W^{1-\varepsilon_w}\right)^{\frac{1}{1-\varepsilon_w}-1}\left[(1-\varepsilon_w)\theta_w W^{-\varepsilon_w} dW_{t-1} + (1-\varepsilon_w)(1-\theta_w)W^{-\varepsilon_W} d\tilde{W}_t\right] \\ &= \left(W^{1-\varepsilon_w}\right)^{\frac{1-(1-\varepsilon_w)}{1-\varepsilon_w}} W^{-\varepsilon_w}\left[\theta_w dW_{t-1} + (1-\theta_w)d\tilde{W}_t\right] \\ &= \left(W^{1-\varepsilon_w}\right)^{\frac{\varepsilon_w}{1-\varepsilon_w}} W^{-\varepsilon_w}\left[\theta_w dW_{t-1} + (1-\theta_w)d\tilde{W}_t\right] \\ &= \theta_w dW_{t-1} + (1-\theta_w)d\tilde{W}_t\end{aligned}.$$

By dividing both sides of the previous expression W_H, we have:

$$\frac{dW_t}{W} = \theta_w\frac{dW_{t-1}}{W} + (1-\theta_w)\frac{d\tilde{W}_t}{W},$$

which can be rewritten as:

$$w_t = \theta_w w_{t-1} + (1-\theta_w)\tilde{w}_t. \text{ (3-1-38)}$$

Eq.(3-1-38) can be rewritten as:

$$\tilde{w}_t = \frac{1}{1-\theta_w} w_t - \frac{\theta_w}{1-\theta_w} w_{t-1}.$$

Plugging the previous expression into Eq. (3-1-37) yields:

$$\frac{1}{1-\theta_w} w_t - \frac{\theta_w}{1-\theta_w} w_{t-1} = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\left[(1+\varepsilon_w\varphi) w_t - \mu_t^W\right] + \beta\theta_w \mathrm{E}_t\left(\frac{1}{1-\theta_w} w_{t+1} - \frac{\theta_w}{1-\theta_w} w_t\right)',$$

Which can be rewritten as:

$$\begin{aligned}
w_t - \theta_w w_{t-1} &= (1-\beta\theta_w)(1-\theta_w) w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w \\
&\quad + \beta\theta_w \mathrm{E}_t\left(w_{t+1} - \theta_w w_t\right) \\
&= (1-\beta\theta_w) w_t - \theta_w (1-\beta\theta_w) w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w \\
&\quad + \beta\theta_w \mathrm{E}_t\left(w_{t+1}\right) - \beta\theta_w^2 w_t \\
&= (1-\beta\theta_w) w_t - \theta_w w_t + \beta\theta_w^2 w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w \\
&\quad + \beta\theta_w \mathrm{E}_t\left(w_{t+1}\right) - \beta\theta_w^2 w_t \\
&= w_t - \theta_w w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w \\
&\quad + \beta\theta_w \mathrm{E}_t\left(w_{t+1}\right) - \beta\theta_w w_t
\end{aligned}$$

Subtracting $w_{H,t}$ from both sides of the previous expression yields:

$$-\theta_w w_{t-1} = -\theta_w w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w + \beta\theta_w \mathrm{E}_t\left(w_{t+1} - w_t\right),$$

which can be rewritten as:

$$\theta_w\left(w_t - w_{t-1}\right) = \beta\theta_w \mathrm{E}_t\left(w_{t+1} - w_t\right) - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w.$$

By dividing both sides by θ_w, we have:

$\pi_t^w = \beta \mathrm{E}_t\left(\pi_{t+1}^w\right) - \kappa_w \mu_t^w$, (3-1-39) [(26) in the text]

with $\kappa_w \equiv \dfrac{(1-\beta\theta_w)(1-\theta_w)}{\theta_w(1+\varepsilon_w\varphi)}$ where:

$\pi_t^w \equiv w_t - w_{t-1}$ (3-1-40) [(28) in the text]

denotes the wage inflation.

There is a relationship between the changes in the real wage and the gap between the wage inflation and the domestic inflation as follows:

$$\begin{aligned} w_t^r - w_{t-1}^r &= w_t - p_{H,t} - \left(w_{H,t-1} - p_{H,t-1}\right) \\ &= w_t - w_{t-1} - \left(p_{H,t} - p_{H,t-1}\right) \\ &= \pi_t^W - \pi_{H,t} \end{aligned}.$$

3.1.9 UIP

The log-linearized UIP $R_t = \mathrm{E}_t\left(E_{t+1}/E_t\right)R_t^*$ is given by

$r_t = r_t^* + \mathrm{E}_t\left(e_{t+1}\right) - e_t$. (3-1-43).

Note that:

$$\frac{dR_t}{R} = \frac{dR_t^*}{R^*} + \frac{dE_{t+1}}{E} - \frac{dE_t}{E}$$

can be rewritten as:

$$\ln\left(\frac{1+r_t}{1+r}\right) = \ln\left(\frac{1+r_t^*}{1+r}\right) + \ln\left(\frac{E_{t+1}}{E}\right) - \ln\left(\frac{E_t}{E}\right),$$

further:

$$\ln\left(1+r_t\right) - \ln\left(1+r\right) = \ln\left(1+r_t^*\right) - \ln\left(1+r\right) + \ln\left(\frac{E_{t+1}}{E}\right) - \ln\left(\frac{E_t}{E}\right).$$

Finally, we have:

$$\ln\left(1+r_t\right) = \ln\left(1+r_t^*\right) + \ln\left(\frac{E_{t+1}}{E}\right) - \ln\left(\frac{E_t}{E}\right).$$

4 Welfare Cost Function and Welfare Relevant Output Gap

4.1 Deriving the Second-order Approximated Utility Function with Linear Terms

The period utility function is given by Eq.(1-1-85), namely:

$\tilde{U}_{t+k} \equiv \ln C_{t+k} - \frac{1}{1+\varphi}\int_0^1 N_{t+k|t}\left(j\right)^{1+\varphi} dj$. (1-1-85)

In equilibrium,

$$\mathcal{N}_t(j) = N_t(j) = \left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W} N_t,$$

Is applied. Plugging this expression into Eq.(1-1-85) yields:

$$\tilde{U}_{t+k} \equiv \ln C_{t+k} - \frac{1}{1+\varphi} N_{t+k}^{1+\varphi} \int_0^1 \left(\frac{W_{t+k}(j)}{W_{t+k}}\right)^{-\varepsilon_w(1+\varphi)} dj.$$

Let define $\left(\Delta_t^w\right)^{1+\varphi} \equiv \int_0^1 \left(\frac{W_{H,t}(j)}{W_{H,t}}\right)^{-\varepsilon_w(1+\varphi)} dj$. Plugging this definition into the previous expression, we have:

$$\tilde{U}_{H,t+k} \equiv \ln C_{t+k} - \frac{1}{1+\varphi} N_{t+k}^{1+\varphi} \left(\Delta_{t+k}^w\right)^{1+\varphi}. \quad (4\text{-}1\text{-}1)$$

Second-order expansion (percentage deviation from steady state in terms of marginal utility of consumption) of Eq.(4-1) is given by:

$$\begin{aligned}\frac{\tilde{U}_{t+k} - \tilde{U}}{\tilde{U}_C C} &= c_{t+k} + \frac{1}{2} c_{t+k}^2 + \frac{1}{2} \frac{\tilde{U}_{CC}}{\tilde{U}_C} C c_{t+k}^2 + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} \left(n_{t+k} + \frac{1}{2} n_{t+k}^2\right) + \frac{1}{2} \frac{\tilde{U}_{NN}}{\tilde{U}_C} \frac{N^2}{C} n_{t+k}^2 \\ &\quad + \frac{\tilde{U}_Z}{\tilde{U}_C} \frac{1}{C} \ln \Delta_{t+k}^w + o\left(\|\xi\|^3\right)\end{aligned}.$$

Note that $\ln \Delta_{t+k}^W$ is $o\left(\|\xi\|^2\right)$.

Plugging $\tilde{U}_C = C^{-1}$, $\tilde{U}_{CC} = -C^{-2}$, $\tilde{U}_N = -N^{\varphi}$, $\tilde{U}_{NN} = -\varphi N^{\varphi-1}$ and $\tilde{U}_Z = -N^{1+\varphi}$ into the previous expression yields:

$$
\begin{aligned}
\frac{U_t - U}{U_C C} &= c_t + \frac{1}{2}c_t^2 + \frac{1}{2}\frac{\left(-C^{-2}\right)}{C^{-1}}Cc_t^2 + \frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}\left(n_{t+k} + \frac{1}{2}n_{t+k}^2\right) + \frac{1}{2}\frac{\left(-\varphi N^{\varphi-1}\right)}{\tilde{U}_C}\frac{N^2}{C}n_{t+k}^2 \\
&\quad - \frac{N^{1+\varphi}}{\tilde{U}_C}\frac{1}{C}z_{H,t+k}^{W} + o\left(\|\xi\|^3\right) \\
&= c_t + \frac{1}{2}c_t^2 - \frac{1}{2}\frac{C^2}{C^2}c_t^2 + \frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}\left(n_{t+k} + \frac{1}{2}n_{t+k}^2\right) + \varphi\frac{1}{2}\frac{\left(-N^{\varphi}\right)}{\tilde{U}_C}\frac{1}{N}\frac{N^2}{C}n_{t+k}^2 \\
&\quad + \frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}z_{H,t+k}^{W} + o\left(\|\xi\|^3\right) \\
&= c_t + \frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}n_{t+k} + \frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}\frac{1}{2}n_{t+k}^2 + \varphi\frac{1}{2}\frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}n_{t+k}^2 + \frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}\ln\Delta_{t+k}^{W} + o\left(\|\xi\|^3\right) \\
&= c_t + \frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}n_{t+k} + \frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}\frac{1}{2}(1+\varphi)n_{t+k}^2 + \frac{\tilde{U}_N}{\tilde{U}_C}\frac{N}{C}\ln\Delta_{t+k}^{W} + o\left(\|\xi\|^3\right)
\end{aligned}
.
$$

Plugging $-\frac{\tilde{U}_N}{\tilde{U}_C} \equiv 1 - \Phi$ into the previous expression yields:

$$
\begin{aligned}
\frac{\tilde{U}_t - \tilde{U}}{\tilde{U}_C C} &= c_t - (1-\Phi)\frac{N}{C}n_{t+k} - (1-\Phi)\frac{N}{C}\frac{1+\varphi}{2}n_{t+k}^2 - (1-\Phi)\frac{N}{C}\ln\Delta_{t+k}^{w} + o\left(\|\xi\|^3\right) \\
&= c_t + \Phi\frac{N}{C}n_{t+k} - \frac{N}{C}n_{t+k} - (1-\Phi)\frac{N}{C}\frac{1+\varphi}{2}n_{t+k}^2 - (1-\Phi)\frac{N}{C}\ln\Delta_{t+k}^{w} + o\left(\|\xi\|^3\right). \\
&= c_t + \Phi\frac{N}{C}n_{t+k} - \frac{N}{C}\left[n_{t+k} + \frac{(1-\Phi)(1+\varphi)}{2}n_{t+k}^2 + (1-\Phi)\ln\Delta_{t+k}^{w}\right] + o\left(\|\xi\|^3\right)
\end{aligned}
$$

Plugging $n_{t+k} = y_{H,t+k} - a_{t+k} + \ln\Delta_{t+k}^{p}$ into the last line in the previous expression yields:

$$
\begin{aligned}
\frac{\tilde{U}_t - \tilde{U}}{\tilde{U}_C C} &= c_{t+k} + \frac{\Phi}{\sigma_C}\left(y_{t+k} + \ln\Delta_{t+k}^{p}\right) \\
&\quad - \left[\begin{matrix}\left(y_{t+k} + \ln\Delta_{t+k}^{p}\right) \\ + \frac{(1-\Phi)(1+\varphi)}{2}n_{t+k}^2 + (1-\Phi)\ln\Delta_{t+k}^{w}\end{matrix}\right] + o\left(\|\xi\|^3\right) \\
&= c_{t+k} - (1-\Phi)y_{t+k} - \left[\begin{matrix}\frac{(1-\Phi)(1+\varphi)}{2}n_{t+k}^2 \\ + (1-\Phi)\ln\Delta_{t+k}^{p} + (1-\Phi)\ln\Delta_{t+k}^{w}\end{matrix}\right] + o\left(\|\xi\|^3\right)
\end{aligned}
\quad , \text{(4-5-3)}
$$

where we use $N/C = (C/Y)^{-1} = \sigma_C^{-1}$ and $\ln\Delta_{t+k}^{p} = \ln\left[\int_0^1\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}dh\right]$.

4.2 Second-order Approximation of Price and Wage Dispersions

Relative price of good *h* can be approximated as follows:

$$\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon}=1+\left(1-\varepsilon_p\right)\hat{p}_{H,t}(i)+\frac{1}{2}\left(1-\varepsilon_p\right)^2\hat{p}_{H,t}(i)^2+o\left(\left\|\xi\right\|^3\right), \text{(4-5-12)}$$

with $\hat{p}_{H,t}(i)\equiv\ln P_{H,t}(i)-\ln P_{H,t}$.

The price index $P_{H,t}\equiv\left[\int_0^1 P_{H,t}(i)^{1-\varepsilon}di\right]^{\frac{1}{1-\varepsilon}}$ implies:

$$1=\frac{1}{P_{H,t}}\left[\int_0^1 P_{H,t}(i)^{1-\varepsilon_p}di\right]^{\frac{1}{1-\varepsilon_p}}.$$

Hence, we have:

$$\begin{aligned}1^{1-\varepsilon}&=\left(\frac{1}{P_{H,t}}\right)^{1-\varepsilon_p}\int_0^{\tilde{\alpha}}P_t(i)^{1-\varepsilon_p}di\\&=\int_0^1\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon_p}di\qquad,\\&=1\end{aligned}$$

which implies that:

$$\begin{aligned}E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon_p}&=\int_0^1\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon_p}di\\&=1\end{aligned}. \text{(4-5-13)}$$

Eq.(4-5-12) can be rewritten as:

$$\left(1-\varepsilon_p\right)\hat{p}_{H,t}(i)=-\frac{1}{2}\left(1-\varepsilon_p\right)^2\hat{p}_{H,t}(i)^2-\left[1-\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon_p}\right]+o\left(\left\|\xi\right\|^3\right),$$

Taking conditional expectation, The previous expression can be rewritten as:

$$\begin{aligned}E_i\left[\hat{p}_{H,t}(i)\right]&=-\frac{1-\varepsilon_p}{2}E_i\left[\hat{p}_{H,t}(i)^2\right]-\frac{1}{1-\varepsilon_p}\left[1-E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon}\right]+o\left(\left\|\xi\right\|^3\right)\\&=-\frac{1-\varepsilon_p}{2}E_i\left[\hat{p}_{H,t}(i)^2\right]+o\left(\left\|\xi\right\|^3\right)\end{aligned}, \text{(4-5-14)}$$

where we use Eq.(4-5-13) to derive the second line.

Second-order approximation to $\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}$ yields:

$$\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} = \exp\left[-\varepsilon_p \hat{p}_{H,t}(i)\right] \\ = 1 - \varepsilon_p \hat{p}_{H,t}(i) + \frac{\varepsilon^2}{2}\hat{p}_{H,t}(i)^2 + o\left(\|\xi\|^3\right).$$

Taking conditional expectation on the previous expression yields:

$$E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon_p} = 1 - \varepsilon_p \mathrm{E}_i\left[\hat{p}_{H,t}(i)\right] + \frac{\varepsilon_p^2}{2}\mathrm{E}_i\left[\hat{p}_{H,t}(i)^2\right] + o\left(\|\xi\|^3\right). \text{ (4-5-15)}$$

Plugging Eq.(4-5-14) into (4-5-15) yields:

$$\begin{aligned} E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} &= 1 - \varepsilon_p\left\{-\frac{1-\varepsilon}{2}E_h\left[\hat{p}_{H,t}(i)^2\right]\right\} + \frac{\varepsilon_p^2}{2}\mathrm{E}_i\left[\hat{p}_{H,t}(i)^2\right] + o\left(\|\xi\|^3\right) \\ &= 1 + \frac{\varepsilon_p\left[\left(1-\varepsilon_p\right)+\varepsilon_p\right]}{2}\mathrm{E}_i\left[\hat{p}_{H,t}(i)^2\right] + o\left(\|\xi\|^3\right) \\ &= 1 + \frac{\varepsilon_p}{2}\mathrm{E}_i\left[\hat{p}_{H,t}(i)^2\right] + o\left(\|\xi\|^3\right) \\ &= 1 + \frac{\varepsilon_p}{2}\mathrm{var}_i\left[\hat{p}_{H,t}(i)\right] + o\left(\|\xi\|^3\right) \end{aligned} \text{. (4-14-16)}$$

Notice that $E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} = \int_0^1\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di$ and $\Delta_t^p = \int_0^1\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di$. By using these facts, Eq.(4-14-16) can be rewritten as:

$$\begin{aligned} \ln\Delta_t^p &= \ln\left\{1 + \frac{\varepsilon_p}{2}\mathrm{var}_i\left[\hat{p}_{H,t}(i)\right]\right\} + o\left(\|\xi\|^3\right) \\ &\approx \frac{\varepsilon_p}{2}\mathrm{var}_i\left[\hat{p}_{H,t}(i)\right] + o\left(\|\xi\|^3\right) \end{aligned} \text{. (4-2-6)}$$

Further, as shown in Woodford (2003), $z_{H,t}$ can be rewritten as:

$$\sum_{k=0}^{\infty}\beta^{t}\mathrm{E}_t\left(\ln\Delta_t^p\right) = \frac{\varepsilon_p}{2\kappa_p}\sum_{k=0}^{\infty}\beta^{t}\mathrm{E}_t\left(\pi_{H,t}^2\right), \text{ (4-6-2)}$$

as long as Eq.(4-5-10) is applicable.

Relative wage *j* can be approximated as follows:

$$\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W} = 1+(1-\varepsilon_W)\hat{w}_t(j)+\frac{1}{2}(1-\varepsilon_W)^2\hat{w}_t(j)^2+o\left(\|\xi\|^3\right), \text{ (4-6-3)}$$

with $\hat{w}_t(j)\equiv \ln W_t(j)-\ln W_t$.

The wage index $W_t \equiv \left[\int_0^1 W_t(j)^{1-\varepsilon_W}dj\right]^{\frac{1}{1-\varepsilon_W}}$ implies:

$$1=\frac{1}{W_t}\left[\int_0^1 W_t(j)^{1-\varepsilon_W}dj\right]^{\frac{1}{1-\varepsilon_W}}.$$

Hence, we have:

$$\begin{aligned} 1^{1-\varepsilon_W} &= \left(\frac{1}{W_t}\right)^{1-\varepsilon_W}\int_0^1 W_t(j)^{1-\varepsilon_W}dj \\ &= \int_0^1\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W}dj \qquad , \\ &= 1 \end{aligned}$$

which implies that:

$$\begin{aligned} E_j\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W} &= \int_0^1\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W}dj \\ &= 1 \end{aligned} \text{. (4-14-17)}$$

Eq.(4-6-3) can be rewritten as:

$$(1-\varepsilon_W)\hat{w}_t(j)=-\frac{1}{2}(1-\varepsilon_W)^2\hat{w}_t(j)^2-\left[1-\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W}\right]+o\left(\|\xi\|^3\right),$$

Taking conditional expectation, The previous expression can be rewritten as:

$$\begin{aligned} \mathrm{E}_j\left[\hat{w}_t(j)\right] &= -\frac{1-\varepsilon_W}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]-\frac{1}{1-\varepsilon_W}\left[1-\mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W}\right]+o\left(\|\xi\|^3\right) \\ &= -\frac{1-\varepsilon_W}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]+o\left(\|\xi\|^3\right) \end{aligned} \text{, (4-2-10)}$$

where we use Eq.(4-14-17) to derive the second line.

Second-order approximation to $\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W}$ yields:

$$\begin{aligned}\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} &= \exp\left[-\varepsilon_W(1+\varphi)\hat{w}_t(j)\right] \\ &= 1-\varepsilon_W(1+\varphi)\hat{w}_t(j)+\frac{\varepsilon_W^2(1+\varphi)^2}{2}\hat{w}_t(j)^2+o\left(\|\xi\|^3\right)\end{aligned}.$$

Taking conditional expectation on the previous expression yields:

$$\mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} = 1-\varepsilon_W(1+\varphi)\mathrm{E}_j\left[\hat{w}_t(j)\right]+\frac{\varepsilon_W^2(1+\varphi)^2}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]+o\left(\|\xi\|^3\right). \quad (4\text{-}2\text{-}11)$$

Plugging Eq.(4-2-11) into (4-2-10) yields:

$$\begin{aligned}\mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} &= 1-\varepsilon_W(1+\varphi)\left\{-\frac{1-\varepsilon_W}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]\right\}+\frac{\varepsilon_W^2(1+\varphi)^2}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right] \\ &\quad +o\left(\|\xi\|^3\right) \\ &= 1+\frac{\varepsilon_W(1+\varphi)\left[(1-\varepsilon_W)+\varepsilon_W(1+\varphi)\right]}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]+o\left(\|\xi\|^3\right) \quad . (4\text{-}2\text{-}12) \\ &= 1+\frac{\varepsilon_W(1+\varphi)(1+\varepsilon_W\varphi)}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]+o\left(\|\xi\|^3\right) \\ &= 1+\frac{\varepsilon_W(1+\varphi)(1+\varepsilon_W\varphi)}{2}\mathrm{var}_j\left[\hat{w}_t(j)\right]+o\left(\|\xi\|^3\right)\end{aligned}$$

Notice that $\mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} = \int_0^1\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} dj$ and $\left(\Delta_t^w\right)^{1+\varphi} \equiv \int_0^1\left(\frac{W_{H,t}(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} dj$. By using these facts, Eq.(4-2-12) can be rewritten as:

$$
\begin{aligned}
\ln\left(\Delta_t^w\right)^{1+\varphi} &= \ln\int_0^1\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} dj \\
&= \ln \mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} \\
&= \ln\left\{1+\frac{\varepsilon_W(1+\varphi)(1+\varepsilon_W\varphi)}{2}\mathrm{var}_j\left[\hat{w}_t(j)\right]\right\}+o\left(\|\xi\|^3\right) \\
&\approx \frac{\varepsilon_W(1+\varphi)(1+\varepsilon_W\varphi)}{2}\mathrm{var}_j\left[\hat{w}_t(j)\right]+o\left(\|\xi\|^3\right)
\end{aligned}
.
$$

The previous expression can be rewritten as:

$$
\ln\left(\Delta_t^w\right)^{1+\varphi} = \frac{\varepsilon_W(1+\varepsilon_W\varphi)}{2}\mathrm{var}_j\left[\hat{w}_{H,t}(j)\right]+o\left(\|\xi\|^3\right). \text{ (4-2-12)}
$$

Further, as shown in Woodford (2003), $\ln\Delta_{t+k}^w$ can be rewritten as:

$$
\sum_{k=0}^{\infty}\beta^{\mathrm{t}}\mathrm{E}_t\left(\ln\Delta_{t+k}^w\right)=\frac{\varepsilon_W(1+\varepsilon_W\varphi)}{2\kappa_W}\sum_{k=0}^{\infty}\beta^{\mathrm{t}}\mathrm{E}_t\left(\pi_{H,t}^W\right)^2. \text{ (4-2-13)}
$$

Iterating Eq.(4-5-3) yields:

$$
\begin{aligned}
\tilde{\mathcal{W}} = \sum_{k=0}^{\infty}\beta^{\mathrm{t}}\mathrm{E}_t&\left\{c_{t+k}-(1-\Phi)y_{t+k}-\left[\begin{array}{l}\frac{(1-\Phi)(1+\varphi)}{2}n_{t+k}^2 \\ +(1-\Phi)\ln\Delta_{t+k}^p+(1-\Phi)\ln\Delta_{t+k}^w\end{array}\right]\right\}. \\
&+o\left(\|\xi\|^3\right)
\end{aligned}
$$

Plugging Eqs. (4-6-2) and (4-2-13) yields:

$$
\tilde{\mathcal{W}} = \sum_{k=0}^{\infty}\beta^{\mathrm{t}}\mathrm{E}_t\left[\begin{array}{l} c_{t+k}-(1-\Phi)y_{t+k}-\frac{(1-\Phi)(1+\varphi)}{2}n_{H,t+k}^2 \\ -\frac{\varepsilon_p(1-\Phi)}{2\kappa_p}\pi_{H,t+k}^2-\frac{\varepsilon_w(1+\varepsilon_w\varphi)(1-\Phi)}{2\kappa_w}\left(\pi_{t+k}^W\right)^2\end{array}\right]+o\left(\|\xi\|^3\right), \text{ (4-2-14)}
$$

with $\tilde{\mathcal{W}}_H \equiv \sum_{k=0}^{\infty}\beta^k\mathrm{E}_t\left(\frac{\tilde{U}_{t+k}-\tilde{U}}{\tilde{U}_C C}\right)$.

Eq.(3-1-8)’ can be rewritten as:

$$
c_t=\frac{1}{1-\upsilon}y_t-\frac{\eta\upsilon(2-\upsilon)}{1-\upsilon}s_t-\frac{\upsilon}{1-\upsilon}z_{1,t}^* \quad \text{(4-2-15)}
$$

Plugging Eqs.(4-2-15) into Eq.(4-2-14) yields:

$$\tilde{\mathcal{W}} = \sum_{k=0}^{\infty} \beta^t \mathrm{E}_t \left\{ \begin{array}{l} \Phi y_{t+k} - \frac{\eta\upsilon(2-\upsilon)}{1-\upsilon} s_{t+k} - \frac{(1-\Phi)(1+\varphi)}{2} n_{H,t+k}^2 \\ -\frac{\varepsilon_p(1-\Phi)}{2\kappa_p} \pi_{H,t+k}^2 - \frac{\varepsilon_w(1+\varepsilon_w\varphi)(1-\Phi)}{2\kappa_w} \left(\pi_{t+k}^W\right)^2 \end{array} \right\} + o\left(\|\xi\|^3\right), \text{ (4-2-17)}$$

which is applicable for the SOE withOUT default risk.

4.3 Second-order Approximation of the AS Equation

Second-order approximated AS equation is given by:

$$\bar{\nu} = \kappa_p \sum_{k=0}^{\infty} \beta^k \mathrm{E}_t \left\{ \begin{array}{l} w_{t+k}^r - \frac{2\tau}{1-\tau}\hat{\tau}_{t+k} + \frac{\tau}{1-\tau} c_{t+k} - \frac{\tau}{1-\tau} y_{t+k} - \frac{\tau}{1-\tau} x_{H,t+k} + \frac{1}{2}\left(w_{t+k}^r\right)^2 \\ + \frac{\tau}{2(1-\tau)} c_{t+k}^2 - \frac{\tau}{2(1-\tau)} y_{t+k}^2 - \frac{\tau}{2(1-\tau)} x_{H,t+k}^2 - w_{t+k}^r a_{t+k} - c_{t+k} w_{t+k}^r \\ + c_{t+k} a_{t+k} + y_{t+k} w_{t+k}^r - y_{t+k} a_{H,t+k} + x_{t+k} w_{H,t+k}^r - x_{H,t+k} a_{t+k} \end{array} \right.$$

$$+ \frac{\kappa_p \varepsilon_p}{2} \sum_{k=0}^{\infty} \beta^k \pi_{H,t+k}^2 + \text{s.o.t.i.p.} + o\left(\|\xi\|^3\right)$$

. (4-3-1)

4.4 Second-order Approximation of the Wage Equation

Eq.(1-1-95) can be rewritten as:

$$\frac{\tilde{W}_{H,t}}{W_{H,t}} = \frac{\mathcal{M}^W \sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left(MRS_{H,t+k|t} N_{H,t+k|t} C_{t+k}^{-1} \right)}{\sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left[N_{H,t+k|t} \left(P_{t+k} C_{t+k}\right)^{-1} \right] W_{H,t}}. \text{ (4-4-1)}$$

Let define:

$$K_{H,t}^W \equiv \mathcal{M}^W \sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left(MRS_{H,t+k|t} N_{H,t+k|t} C_{t+k}^{-1} \right), \text{ (4-4-2)}$$

$$F_{H,t}^W \equiv \sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left[N_{H,t+k|t} \left(P_{t+k} C_{t+k}\right)^{-1} \right] W_{H,t}. \text{ (4-4-3)}$$

By plugging Eqs.(4-4-2) and (4-4-3) into Eq.(4-6-3) yields:

$$\frac{\tilde{W}_{H,t}}{W_{H,t}} = \frac{K_{H,t}^W}{F_{H,t}^W}. \text{ (4-4-4)}$$

Multiplying Eq.(4-4-4) by $W_{H,t}$ yields:

$$\tilde{W}_{H,t} = \frac{W_{H,t} K_{H,t}^W}{F_{H,t}^W}$$

Plugging the previous expression Eq.(1-1-97) yields:

$W_{H,t} = \left[\theta_W W_{H,t-1}^{1-\varepsilon_W} + (1-\theta_W)\left(\frac{W_{H,t}K_{H,t}^W}{F_{H,t}^W}\right)^{1-\varepsilon_W}\right]^{\frac{1}{1-\varepsilon_W}}$, which can be rewritten as:

$$\begin{aligned} W_{H,t}^{1-\varepsilon_W} &= (1-\theta_W)\left(\frac{W_{H,t}K_{H,t}^W}{F_{H,t}^W}\right)^{1-\varepsilon_W} + \theta_W W_{H,t-1}^{1-\varepsilon_W} \\ &= (1-\theta_W) W_{H,t}^{1-\varepsilon_W}\left(K_{H,t}^W\right)^{1-\varepsilon_W}\left(F_{H,t}^W\right)^{-(1-\varepsilon_W)} + \theta_W W_{H,t-1}^{1-\varepsilon_W} \end{aligned}.$$

Dividing the previous expression by $W_{H,t}^{1-\varepsilon_W}$ yields:

$$\begin{aligned} 1 &= (1-\theta_W)\left(K_{H,t}^W\right)^{1-\varepsilon_W}\left(F_{H,t}^W\right)^{-(1-\varepsilon_W)} + \theta_W\left(\frac{W_{H,t-1}}{W_{H,t}}\right)^{1-\varepsilon_W}, \\ &= (1-\theta_W)\left(K_{H,t}^W\right)^{1-\varepsilon_W}\left(F_{H,t}^W\right)^{-(1-\varepsilon_W)} + \theta_W\left(\Pi_{H,t}^W\right)^{-(1-\varepsilon_W)} \end{aligned}$$

which can be rewritten as:

$$\frac{1}{1-\theta_W}\left[1-\theta_W\left(\Pi_{H,t}^W\right)^{-(1-\varepsilon_W)}\right] = \left(\frac{F_{H,t}^W}{K_{H,t}^W}\right)^{\varepsilon_W-1}. \text{ (4-4-6)}$$

Taking logarithm both sides of Eq.(4-4-6) yields:

$$\ln\left[\frac{1}{1-\theta_W} - \frac{\theta_W}{1-\theta_W}\left(\Pi_{H,t}^W\right)^{-(1-\varepsilon_W)}\right] = (\varepsilon_W - 1)\left(\ln F_{H,t}^W - \ln K_{H,t}^W\right),$$

which can be rewritten as:

$$-\ln\left[\frac{1}{1-\theta_W} - \frac{\theta_W}{1-\theta_W}\left(\Pi_{H,t}^W\right)^{-(1-\varepsilon_W)}\right] = (\varepsilon_W - 1)\left(\ln K_{H,t}^W - \ln F_{H,t}^W\right). \text{ (4-4-7)}$$

Second-order approximation of the LHS of Eq.(4-4-6) is given by:

$$-\ln\left[\frac{1}{1-\theta_W} - \frac{\theta_W}{1-\theta_W}\left(\Pi_{H,t}^W\right)^{-(1-\varepsilon_W)}\right] = -\frac{\theta_W(\varepsilon_W-1)}{1-\theta_W}\left[\pi_{H,t}^W + \frac{\varepsilon_W-1}{2(1-\theta_W)}\left(\pi_{H,t}^W\right)^2\right] + o\left(\|\xi\|^3\right).$$

Eq.(4-4-3) can be rewritten as:

$$
\begin{aligned}
F_{H,t}^{W} &= W_{H,t}\left[N_{H,t|t}\left(P_t C_t\right)^{-1} + \beta\theta_W N_{H,t+1|t}\left(P_{t+1}C_{t+1}\right)^{-1} + \left(\beta\theta_W\right)^2 N_{H,t+2|t}\left(P_{t+2}C_{t+2}\right)^{-1} + \cdots\right] \\
&= W_{H,t}\left[\begin{array}{l} \left(\dfrac{\tilde{W}_t}{W_{H,t}}\right)^{-\varepsilon_W} N_{H,t+k}\left(P_t C_t\right)^{-1} + \beta\theta_W \left(\dfrac{\tilde{W}_t}{W_{H,t+1}}\right)^{-\varepsilon_W} N_{H,t+k}\left(P_{t+1}C_{t+1}\right)^{-1} \\ + \left(\beta\theta_W\right)^2 \left(\dfrac{\tilde{W}_t}{W_{H,t+2}}\right)^{-\varepsilon_W} N_{H,t+k}\left(P_{t+2}C_{t+2}\right)^{-1} + \cdots \end{array}\right] \\
&= \left[\begin{array}{l} C_t^{-1}\left(\dfrac{\tilde{W}_{H,t}}{W_{H,t}}\right)^{-\varepsilon_W} N_{H,t}\dfrac{P_{H,t}}{P_t}\dfrac{W_{H,t}}{P_{H,t}} + \beta\theta_W C_{t+1}^{-2}\left(\dfrac{\tilde{W}_{H,t}}{W_{H,t}}\dfrac{W_{H,t}}{W_{H,t+1}}\right)^{-\varepsilon_W} N_{H,t+k}\dfrac{W_{H,t}}{W_{H,t+1}}\dfrac{W_{H,t+1}}{P_{H,t+1}}\dfrac{P_{H,t+1}}{P_{t+1}} \\ + \left(\beta\theta_W\right)^2 C_{t+2}^{-1}\left(\dfrac{\tilde{W}_{H,t}}{W_{H,t}}\dfrac{W_{H,t}}{W_{H,t+1}}\dfrac{W_{H,t+1}}{W_{H,t+2}}\right)^{-\varepsilon_W} N_{H,t+k}\dfrac{W_{H,t}}{W_{H,t+1}}\dfrac{W_{H,t+1}}{W_{H,t+2}}\dfrac{W_{H,t+2}}{P_{H,t+2}}\dfrac{P_{H,t+2}}{P_{t+2}} + \cdots \end{array}\right], \\
&= \left[\begin{array}{l} C_t^{-1}\left(\tilde{X}_{H,t}^{W}\right)^{-\varepsilon_W} N_{H,t}X_{H,t}W_{H,t}^{r} + \beta\theta_W C_{t+1}^{-1}\left(\tilde{X}_{H,t}^{W}\right)^{-\varepsilon_W}\left(\Pi_{H,t+1}^{W}\right)^{\varepsilon_W-1} N_{H,t+k}X_{H,t+1}W_{H,t+1}^{r} \\ + \left(\beta\theta_W\right)^2 C_{t+1}^{-1}\left(\tilde{X}_{H,t}^{W}\right)^{-\varepsilon_W}\left(\Pi_{H,t+1}^{W}\Pi_{H,t+2}^{W}\right)^{\varepsilon_W-1} N_{H,t+2}X_{H,t+2}W_{H,t+2}^{r} + \cdots \end{array}\right]
\end{aligned}
$$

with $\tilde{X}_{H,t}^{W} \equiv \dfrac{\tilde{W}_t}{W_{H,t}}$, $W_{H,t}^{r} \equiv \dfrac{W_{H,t}}{P_{H,t}}$ and $\Pi_{H,t}^{W} \equiv \dfrac{W_{H,t}}{W_{H,t-1}}$ where we use Eq.(1-1-89). The last line can be rewritten as compact form as:

$$
F_{H,t}^{W} = \sum_{k=0}^{\infty}\left(\beta\theta_W\right)^k \mathrm{E}_t\left[\left(\tilde{X}_{H,t}^{W}\right)^{-\varepsilon_W} C_{t+k}^{-1}N_{H,t+k}X_{H,t+k}W_{H,t+k}^{r}\left(\prod_{s=1}^{k}\Pi_{H,t+s}^{W}\right)\right]^{\varepsilon_W-1}.
$$

Eq.(4-4-2) can be rewritten as:

$$
\begin{aligned}
K_{H,t}^{W} &= \mathcal{M}^{W}\left[MRS_{H,t|t}N_{H,t|t}C_t^{-1} + \beta\theta_W MRS_{H,t+1|t}N_{H,t+1|t}C_{t+1}^{-1} + \left(\beta\theta_W\right)^k MRS_{H,t+2|t}N_{H,t+2|t}C_{t+2}^{-1} + \cdots\right] \\
&= \mathcal{M}^{W}\left[N_{H,t|t}^{1+\varphi} + \beta\theta_W N_{H,t+1|t}^{1+\varphi} + \left(\beta\theta_W\right)^k N_{H,t+2|t}^{1+\varphi} + \cdots\right] \\
&= \mathcal{M}^{W}\left[\left(\dfrac{\tilde{W}_{H,t}}{W_{H,t}}\right)^{-\varepsilon_W(1+\varphi)} N_{H,t}^{1+\varphi} + \beta\theta_W\left(\dfrac{\tilde{W}_{H,t}}{W_{H,t+1}}\right)^{-\varepsilon_W(1+\varphi)} N_{H,t+1}^{1+\varphi} + \left(\beta\theta_W\right)^2\left(\dfrac{\tilde{W}_{H,t}}{W_{H,t+2}}\right)^{-\varepsilon_W(1+\varphi)} N_{H,t+2}^{1+\varphi} + \cdots\right] \\
&= \mathcal{M}^{W}\left[\begin{array}{l} \left(\dfrac{\tilde{W}_{H,t}}{W_{H,t}}\right)^{-\varepsilon_W(1+\varphi)} N_{H,t}^{1+\varphi} + \beta\theta_W\left(\dfrac{\tilde{W}_{H,t}}{W_{H,t}}\dfrac{W_{H,t}}{W_{H,t+1}}\right)^{-\varepsilon_W(1+\varphi)} N_{H,t+1}^{1+\varphi} \\ + \left(\beta\theta_W\right)^2\left(\dfrac{\tilde{W}_{H,t}}{W_{H,t}}\dfrac{W_{H,t}}{W_{H,t+1}}\dfrac{W_{H,t+1}}{W_{H,t+2}}\right)^{-\varepsilon_W(1+\varphi)} N_{H,t+2}^{1+\varphi} + \cdots \end{array}\right] \\
&= \mathcal{M}^{W}\left[\begin{array}{l} \left(\tilde{X}_{H,t}^{W}\right)^{-\varepsilon_W(1+\varphi)} N_{H,t}^{1+\varphi} + \beta\theta_W\left(\tilde{X}_{H,t}^{W}\right)^{-\varepsilon_W(1+\varphi)}\left(\Pi_{H,t+1}^{W}\right)^{\varepsilon_W(1+\varphi)} N_{H,t+1}^{1+\varphi} \\ + \left(\beta\theta_W\right)^2\left(\tilde{X}_{H,t}^{W}\right)^{-\varepsilon_W(1+\varphi)}\left(\Pi_{H,t+1}^{W}\Pi_{H,t+2}^{W}\right)^{\varepsilon_W(1+\varphi)} N_{H,t+2}^{1+\varphi} + \cdots \end{array}\right]
\end{aligned}
$$

, which can be rewritten as:

$$K_{H,t}^{W} = M^{W}\sum_{k=0}^{\infty}\left(\beta\theta_{W}\right)^{k}\left(\tilde{X}_{H,t}^{W}\right)^{-\varepsilon_{W}(1+\varphi)} N_{H,t+k}^{1+\varphi}\left(\prod_{s=1}^{k}\Pi_{H,t+s}^{W}\right)^{\varepsilon_{W}(1+\varphi)}$$

Finally, second-order approximated wage equation is given by:

$$\bar{\nu}^{w} = \kappa_{w}\sum_{k=0}^{\infty}\beta^{k}\mathrm{E}_{t}\begin{bmatrix}\varphi n_{t+k} + c_{t+K} - x_{H,t+k} - w_{H,t+k}^{r} + \dfrac{\varphi(2+\varphi)}{2}n_{t+k}^{2} - \dfrac{1}{2}c_{t+k}^{2} \\ -\dfrac{1}{2}x_{t+k}^{2} - \dfrac{1}{2}\left(w_{t+k}^{r}\right)^{2} + c_{t+k}n_{t+k} + c_{t+k}x_{H,t+k} + c_{t+k}w_{t+k}^{r} \\ -n_{t+k}x_{H,t+k} - n_{t+k}w_{H,t+k}^{r} - x_{t+k}w_{H,t+k}^{r} + \dfrac{\varepsilon_{w}(1+\varphi)}{2}\left(\pi_{H,t+k}^{W}\right)^{2}\end{bmatrix}. \text{ (4-4-8)}$$

4.7 Second-order Approximation of the Market Clearing Condition

Eq.(1-2-76) can be rewritten as:

$$Y_{t} = \left(1-\upsilon\right)X_{H,t}^{-\eta}C_{t} + \upsilon S_{t}^{\eta}Z_{1,t}^{*}. \text{ (4-7-1)}$$

Where we use $X_{H,t} \equiv P_{H,t}/P_{t}$ and Eq.(1-1-20).

Second-order approximation of Eq.(4-7-1) is given by:

$$\begin{aligned}Y_{H}\begin{pmatrix}X_{H,t}, C_{t}, S_{t}, \\ Z_{1,t}^{*}, G_{t}\end{pmatrix} &= Y + Y_{X}\left(X_{H,t}-1\right) + Y_{C}\left(C_{t}-C\right) + Y_{S}\left(S_{t}-1\right) + \frac{1}{2}Y_{XX}\left(X_{H,t}-1\right)^{2} \\ &\quad + \frac{1}{2}Y_{SS}\left(S_{t}-1\right)^{2} + Y_{XC}\left(X_{H,t}-1\right)\left(C_{t}-C\right) + Y_{SZ}\left(S_{t}-1\right)\left(Z_{1,t}^{*}-1\right) \\ &\quad + \text{t.i.p.} + o\left(\|\xi\|^{3}\right) \\ &= Y - \eta(1-\upsilon)C\left(x_{H,t} + \frac{1}{2}x_{H,t}^{2}\right) + (1-\upsilon)C\left(c_{t} + \frac{1}{2}c_{t}^{2}\right) + \upsilon\eta Z_{1}^{*}\left(s_{t} + \frac{1}{2}s_{t}^{2}\right) \\ &\quad + \frac{1}{2}\eta(\eta+1)(1-\upsilon)Cx_{H,t}^{2} + \frac{1}{2}\upsilon\eta(\eta-1)Z_{1}^{*}s_{t}^{2} - \eta(1-\upsilon)Cx_{H,t}c_{t} \\ &\quad + \upsilon\eta Z_{1}^{*}s_{t}z_{1,t}^{*} + \text{t.i.p.} + o\left(\|\xi\|^{3}\right)\end{aligned},$$

which can be rewritten as:

$$
\begin{aligned}
\frac{Y_t-Y}{Y} &= -\eta(1-\upsilon)\frac{C}{Y}\left(x_{H,t}+\frac{1}{2}x_{H,t}^2\right)+(1-\upsilon)\frac{C}{Y}\left(c_t+\frac{1}{2}c_t^2\right)+\upsilon\eta\frac{C}{Y}\left(s_t+\frac{1}{2}s_t^2\right)\\
&\quad+\frac{1}{2}\eta(\eta+1)(1-\upsilon)\frac{C}{Y}x_{H,t}^2+\frac{1}{2}\upsilon\eta(\eta-1)\frac{C}{Y}s_t^2-\eta(1-\upsilon)\frac{C}{Y}x_{H,t}c_t\\
&\quad+\upsilon\eta\frac{C}{Y}s_t z_{1,t}^*+\text{t.i.p.}+o\left(\|\xi\|^3\right)\\
&= -\eta(1-\upsilon)\frac{C}{Y}x_{H,t}+(1-\upsilon)\frac{C}{Y}c_t+\upsilon\eta\frac{C}{Y}s_t-\frac{\eta(1-\upsilon)}{2}\frac{C}{Y}\left[1-(\eta+1)\right]x_{H,t}^2\\
&\quad+\frac{1-\upsilon}{2}\frac{C}{Y}c_t^2+\frac{\upsilon\eta}{2}\frac{C}{Y}\left[1+(\eta-1)\right]s_t^2-\eta(1-\upsilon)\frac{C}{Y}x_{H,t}c_t\\
&\quad+\upsilon\eta\frac{C}{Y}s_t z_{1,t}^*+\text{t.i.p.}+o\left(\|\xi\|^3\right)\\
&= -\eta(1-\upsilon)\frac{C}{Y}x_{H,t}+(1-\upsilon)\frac{C}{Y}c_t+\upsilon\eta\frac{C}{Y}s_t+\frac{\eta^2(1-\upsilon)}{2}\frac{C}{Y}x_{H,t}^2\\
&\quad+\frac{1-\upsilon}{2}\frac{C}{Y}c_t^2+\frac{\upsilon\eta^2}{2}\frac{C}{Y}s_t^2-\eta(1-\upsilon)\frac{C}{Y}x_{H,t}c_t+\upsilon\eta\frac{C}{Y}s_t z_{1,t}^*\\
&\quad+\text{t.i.p.}+o\left(\|\xi\|^3\right)
\end{aligned}
.
$$

Then, we have:

$$
\begin{aligned}
y_t &= -\eta(1-\upsilon)\sigma_C x_{H,t}+(1-\upsilon)\sigma_C c_t+\upsilon\eta\sigma_C s_t+\frac{\eta^2(1-\upsilon)\sigma_C}{2}x_{H,t}^2+\frac{(1-\upsilon)\sigma_C}{2}c_t^2\\
&\quad+\frac{\upsilon\eta^2\sigma_C}{2}s_t^2-\eta(1-\upsilon)\sigma_C x_{H,t}c_t+\upsilon\eta\sigma_C s_t z_{1,t}^*+\text{t.i.p.}+o\left(\|\xi\|^3\right)
\end{aligned}
.
$$

Iterating the previous expression yields:

$$
0=\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\begin{Bmatrix}-y_t-\eta(1-\upsilon)x_{H,t}+(1-\upsilon)c_t+\upsilon\eta s_t\\ +\frac{\eta^2(1-\upsilon)}{2}x_{H,t}^2+\frac{(1-\upsilon)}{2}c_t^2\\ +\frac{\upsilon\eta^2\sigma_C}{2}s_t^2-\eta(1-\upsilon)x_{H,t}c_t+\upsilon\eta\sigma_C s_t z_{1,t}^*\end{Bmatrix}+\text{t.i.p.}+o\left(\|\xi\|^3\right). \quad (4\text{-}7\text{-}2)
$$

4.8 Second-order Approximation of the Definition of the Relative Price

Definition of the relative price of PPI in the SOE is given by:

$$
X_{H,t}\equiv\frac{P_{H,t}}{P_t},
$$

which can be rewritten as:

$$
\begin{aligned}
X_{H,t} &\equiv \frac{P_{H,t}}{P_t} \\
&= \frac{1}{P_t}\frac{P_{H,t}}{P_{F,t}}P_{F,t} \; . \\
&= \frac{1}{P_t}\frac{P_{F,t}}{S_t}
\end{aligned}
$$

By raising both sides on the previous expression the $\eta - 1$ th power yields:

$$
\begin{aligned}
X_{H,t}^{\eta-1} &= \left(\frac{1}{P_t}\frac{P_{F,t}}{S_t}\right)^{\eta-1} \\
&= P_t^{1-\eta}P_{F,t}^{\eta-1}S_t^{1-\eta} \\
&= \left\{\left[(1-\upsilon)P_{H,t}^{1-\eta} + \upsilon P_{F,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}\right\}^{1-\eta} P_{F,t}^{\eta-1}S_t^{1-\eta} \\
&= \left[(1-\upsilon)P_{H,t}^{1-\eta} + \upsilon P_{F,t}^{1-\eta}\right]P_{F,t}^{\eta-1}S_t^{1-\eta} \qquad . \\
&= (1-\upsilon)P_{H,t}^{1-\eta}P_{F,t}^{\eta-1}S_t^{1-\eta} + \upsilon P_{F,t}^{1-\eta}P_{F,t}^{\eta-1}S_t^{1-\eta} \\
&= (1-\upsilon)\left(\frac{P_{F,t}}{P_{H,t}}\right)^{\eta-1} S_t^{-(\eta-1)} + \upsilon P_{F,t}^{1-\eta}P_{F,t}^{-(\eta-1)}S_t^{1-\eta} \\
&= (1-\upsilon)S_t^{\eta-1}S_t^{-(\eta-1)} + \upsilon S_t^{1-\eta} \\
&= (1-\upsilon) + \upsilon S_t^{1-\eta}
\end{aligned}
$$

By raising on the both sides of the previous expression $1/(\eta-1)$ th power yields:

$$
X_{H,t} = \left[(1-\upsilon) + \upsilon S_t^{1-\eta}\right]^{\frac{1}{\eta-1}} . \quad \text{(4-8-1)}
$$

Second-order approximation of Eq.(4-14-24) is given by:

$$
\begin{aligned}
X_{H,t} &= 1 + X_{H\,S}(S_t - 1) + \frac{1}{2}X_{H\,SS}(S_t - 1)^2 + o\left(\|\xi\|^3\right) \\
&= 1 + X_{H\,S}\left(s_t + \frac{1}{2}s_t^2\right) + \frac{1}{2}X_{H\,SS}s_t^2 + o\left(\|\xi\|^3\right) \qquad . \text{(4-8-2)} \\
&= 1 + X_{H\,S}s_t + \frac{X_{H\,S} + X_{H\,SS}}{2}s_t^2 + o\left(\|\xi\|^3\right)
\end{aligned}
$$

$X_{H\,S}$ and $X_{H\,SS}$ are given by:

$$
\begin{aligned}
X_{H\,S} &= \frac{1}{\eta-1}\left[(1-\upsilon)+\upsilon S^{-(\eta-1)}\right]^{\frac{1}{\eta-1}-1}\upsilon\left[-(\eta-1)\right]S^{-\eta} \\
&= \frac{1}{\eta-1}\left[(1-\upsilon)+\upsilon\right]^{\frac{1}{\eta-1}-1}\upsilon\left[-(\eta-1)\right] \\
&= \frac{\upsilon\left[-(\eta-1)\right]}{\eta-1} \\
&= -\upsilon
\end{aligned},
$$

$$
\begin{aligned}
X_{H\,SS} &= \frac{1}{\eta-1}\left(\frac{1}{\eta-1}-1\right)\left[(1-\upsilon)+\upsilon S^{-(\eta-1)}\right]^{\frac{1}{\eta-1}-2}\upsilon\left[-(\eta-1)\right]S^{-\eta}\upsilon\left[-(\eta-1)\right]S^{-\eta} \\
&\quad +\frac{1}{\eta-1}\left[(1-\alpha)+\upsilon S^{-(\eta-1)}\right]^{\frac{1}{\eta-1}-1}\upsilon(-\eta)\left[-(\eta-1)\right]S^{-\eta-1} \\
&= \frac{1}{\eta-1}\frac{1-(\eta-1)}{\eta-1}\left[(1-\upsilon)+\upsilon\right]^{\frac{1}{\eta-1}-2}\upsilon\left[-(\eta-1)\right]\upsilon\left[-(\eta-1)\right] \\
&\quad +\frac{1}{\eta-1}\left[(1-\upsilon)+\upsilon\right]^{\frac{1}{\eta-1}-1}\upsilon\eta(\eta-1) \\
&= \frac{-\eta}{(\eta-1)^2}\upsilon^2(\eta-1)^2+\frac{1}{\eta-1}\upsilon\eta(\eta-1) \\
&= -\eta\upsilon^2+\upsilon\eta \\
&= \upsilon\eta(1-\upsilon)
\end{aligned}.
$$

Plugging those expression into Eq.(4-8-2) yields:

$$
\begin{aligned}
x_{H,t} &= -\upsilon s_t+\frac{(-\upsilon)+\upsilon\eta(1-\upsilon)}{2}s_t^2+o\left(\|\xi\|^3\right) \\
&= -\upsilon s_t+\frac{\upsilon\eta(1-\upsilon)-\upsilon}{2}s_t^2+o\left(\|\xi\|^3\right) \\
&= -\upsilon s_t+\frac{\upsilon\left[\eta(1-\upsilon)-1\right]}{2}s_t^2+o\left(\|\xi\|^3\right)
\end{aligned}.
$$

Iterating the previous expression yields:

$$
0=\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{-x_{H,t}-\upsilon s_t+\frac{\upsilon\left[\eta(1-\upsilon)-1\right]}{2}s_t^2\right\}+o\left(\|\xi\|^3\right). \quad (4\text{-}8\text{-}3)
$$

4.9 Second-order Approximation of the International Risk Sharing Condition

Eq.(1-1-21) can be rewritten as:

$$
Q_t=C_t\frac{Z_{2,t}^*}{Z_t}. \quad (4\text{-}9\text{-}1)
$$

Second-order approximation of Eq.(4-9-1) is given by:

$$
\begin{aligned}
Q_t &= f\left(C_t, Z_t, Z_{2,t}^*\right) \\
&= 1 + f_C\left(C_t - C\right) + f_{CZ_2^*}\left(C_t - C\right)\left(Z_{2,t}^* - 1\right) + f_{CZ}\left(C_t - C\right)\left(Z_t - 1\right) \\
&\quad + \text{t.i.p.} + o\left(\|\xi\|^3\right) \qquad , \\
&= 1 + C\left(c_t + \frac{1}{2}c_t^2\right) + Cc_t z_{2,t}^* - Cc_t z_t + \text{t.i.p.} + o\left(\|\xi\|^3\right) \\
&= 1 + Cc_t + C\frac{1}{2}c_t^2 + Cc_t z_{2,t}^* - Cc_t z_t + \text{t.i.p.} + o\left(\|\xi\|^3\right)
\end{aligned}
$$

which can be rewritten as:

$$q_t = c_t + \frac{1}{2}c_t^2 + c_t z_{2,t}^* - c_t z_t + \text{t.i.p.} + o\left(\|\xi\|^3\right)$$

By Plugging Eq.(3-1-4) into the previous expression, we have:

$$(1-\alpha)s_t = c_t + \frac{1}{2}c_t^2 + c_t z_{2,t}^* - c_t z_t + \text{t.i.p.} + o\left(\|\xi\|^3\right). \quad (4\text{-}9\text{-}2)$$

In the first order, Eq.(4-9-2) can be rewritten as:

$$(1-\alpha)s_t = c_t + \text{t.i.p.} + o\left(\|\xi\|^2\right)$$

Iterating the previous expression can be rewritten as:

$$0 = \sum_{t=0}^{\infty} \beta^t \mathrm{E}_0\left[c_t - (1-\alpha)s_t\right] + \text{t.i.p.} + o\left(\|\xi\|^2\right). \quad (4\text{-}9\text{-}3)$$

4.12 Second-order Approximation of the Production Function

Eq.(1-1-80) can be rewritten as:

$$N_t = \frac{Y_t \Delta_t^p}{A_t}$$

Second-order Approximation of the previous expression is given by:

$$
\begin{aligned}
N_t &= f\left(Y_t, \Delta_t^p, A_t\right) \\
&= N + f_y\left(Y_t - Y\right) + f_\Delta\left(\Delta_t^p - 1\right) + f_{YA}\left(Y_t - Y\right)\left(A_t - 1\right) + \text{t.i.p.} + o\left(\|\xi\|^2\right) \\
&= N + Y\left(y_t + \frac{1}{2}y_t^2\right) + Y\ln\Delta_t^p - Yy_t a_t \qquad , \\
&\quad + \text{t.i.p.} + o\left(\|\xi\|^2\right)
\end{aligned}
$$

which can be rewritten as:

$$\frac{N_t - N}{N} = Y^{-1}\left[Y\left(y_t + \frac{1}{2}y_t^2\right) + Y\ln\Delta_t^p - Yy_t a_t\right]$$
$$+\text{t.i.p.} + o\left(\|\xi\|^2\right) \qquad .$$
$$= \left(y_t + \frac{1}{2}y_t^2\right) + \ln\Delta_t^p - y_t a_t + \text{t.i.p.} + o\left(\|\xi\|^2\right)$$

Further:

$$n_t = y_t + \frac{1}{2}y_t^2 + \ln\Delta_t^p - y_t a_t + \text{t.i.p.} + o\left(\|\xi\|^2\right)$$

Iterating the previous expression yields:

$$0 = \sum_{t=0}^{\infty}\beta^t\left(-n_t + y_t + \frac{1}{2}y_t^2 + \frac{\varepsilon_p}{2\kappa_p}\pi_{H,t}^2 - y_t a_t\right). \text{ (4-12-1)}$$

4.13 Eliminating the Linear-terms

4.13.1 Undetermined Coefficients

In the first-order, Eqs.(4-2-17), (4-5-16), (4-3-1), (4-7-2), (4-8-3), (4-9-3), (4-4-8) and (4-12-1) are given by:

$$\tilde{\mathcal{W}} = \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left[\frac{\Phi}{1-\alpha}y_{t+k} - \frac{\eta\upsilon(2-\upsilon)}{1-\upsilon}s_{t+k}\right] + o\left(\|\xi\|^3\right), \text{ (4-13-1)} \leftarrow \text{(4-2-17)}$$

$$\bar{\nu}_H = \kappa\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(w_t^r + \frac{\tau}{1-\tau}c_t - \frac{\tau}{1-\tau}y_t - \frac{\tau}{1-\tau}x_{H,t}\right) + \text{t.i.p.} + o\left(\|\xi\|^2\right), \text{ (4-13-3)} \leftarrow \text{(4-3-1)}$$

$$0 = \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left[-y_t - \eta(1-\upsilon)x_{H,t} + (1-\upsilon)c_t + \upsilon\eta s_t\right] + \text{t.i.p.} + o\left(\|\xi\|^2\right) \text{ (4-13-4)} \leftarrow \text{(4-7-2)}$$

$$0 = \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(-x_{H,t} - \upsilon s_t\right) + o\left(\|\xi\|^2\right) \text{ (4-13-5)} \leftarrow \text{(4-8-3)}$$

$$0 = \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left[c_t - (1-\upsilon)s_t\right] + \text{t.i.p.} + o\left(\|\xi\|^3\right) \text{ (4-13-6)} \leftarrow \text{(4-9-3)}$$

$$\bar{\nu}^w = \kappa_W\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\varphi n_t + c_t - x_{H,t} - w_t^r\right) + \text{t.i.p.} + o\left(\|\xi\|^2\right), \text{ (4-13-9)} \leftarrow \text{(4-4-8)}$$

$$0 = \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(-n_t + y_t\right). \text{ (4-13-10)} \leftarrow \text{(4-12-1)}$$

How Eq.(4-13-1) corresponds to Eqs.(4-13-2)--(4-13-6), (4-13-9) and (4-13-10). Let solve this problem by method of undetermined coefficients.

The system of coefficients of Eqs.(4-13-2)--(4-13-6), (4-13-9) and (4-13-10) is given by:

,

$$\begin{bmatrix} \Phi \\ -\dfrac{\upsilon\eta(2-\upsilon)}{1-\upsilon} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\dfrac{\tau}{1-\tau}\Theta_2-\Theta_3+\Theta_9 \\ \upsilon\eta\Theta_3-\upsilon\Theta_4-(1-\upsilon)\Theta_5 \\ -\dfrac{\tau}{1-\tau}\Theta_2-\eta(1-\upsilon)\Theta_3-\Theta_4-\Theta_8 \\ \dfrac{\tau}{1-\tau}\Theta_2+(1-\upsilon)\Theta_3+\Theta_5+\Theta_8 \\ \Theta_2-\Theta_8 \\ \varphi\Theta_8-\Theta_9 \end{bmatrix} \begin{matrix} y_t \\ s_t \\ x_{H,t} \\ c_t \\ w^r_{H,t} \\ n_{H,t} \end{matrix}$$

where the LHS is the vector of coefficients of $y_{H,t}$, $\hat{\tau}_{H,t}$, s_t, $x_{H,t}$, c_t, $w^r_{H,t}$ and $n_{H,t}$ in Eq.(4-14-1) and the Θ_1, Θ_2, Θ_3, Θ_4, Θ_5, Θ_8 and Θ_9 are undetermined coefficients on Eqs. (4-14-2)—(4-14-10), repectively. The previous expression can be rewritten as:

$$\begin{aligned} &\Phi=-\frac{\tau}{1-\tau}\Theta_2-\Theta_3+\Theta_9 \\ &\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}=-\upsilon\eta\Theta_3+\upsilon\Theta_4+(1-\upsilon)\Theta_5 \\ &0=-\frac{\tau}{1-\tau}\Theta_2-\eta(1-\upsilon)\Theta_3-\Theta_4-\Theta_8 \qquad \text{(4-13-11)} \\ &0=\frac{\tau}{1-\tau}\Theta_2+(1-\upsilon)\Theta_3+\Theta_5+\Theta_8 \\ &0=\Theta_2-\Theta_8 \\ &0=\varphi\Theta_8-\Theta_9 \end{aligned}$$

The 6th equality in Eq.(4-13-11) can be rewritten as:

$\Theta_9=\varphi\Theta_8$. (4-13-12)

The 5th equality in Eq.(4-14-11) can be rewritten as:

$\Theta_8=\Theta_2$. (4-13-13)

The 4th equality in Eq.(4-13-11) can be rewritten as:

$$\Theta_5=-\frac{\tau}{1-\tau}\Theta_2-(1-\upsilon)\Theta_3-\Theta_8 \text{ . (4-13-16)'}$$

The 3th equality in Eq.(4-13-11) can be rewritten as:

$$\Theta_4=-\frac{\tau}{1-\tau}\Theta_2-\eta(1-\upsilon)\Theta_3-\Theta_8 \text{ . (4-13-17)'}$$

The 2rd equality in Eq.(4-13-11) can be rewritten as:

$$\Theta_3=-\frac{2-\upsilon}{1-\upsilon}+\frac{1}{\eta}\Theta_4+\frac{1-\upsilon}{\upsilon\eta}\Theta_5 \text{ . (4-13-18)}$$

The 1st equality in Eq.(4-13-11) can be rewritten as:

$$\Theta_2=-\frac{(1-\tau)\Phi}{\tau}-\Theta_3+\frac{1-\tau}{\tau}\Theta_9 \quad \text{(4-13-20)'}$$

Plugging Eqs.(4-13-12), (4-13-13) into Eq.(4-13-16)' yields:

$$\begin{aligned}\Theta_5&=-\frac{\tau+1-\tau}{1-\tau}\Theta_2-(1-\upsilon)\Theta_3\\&=-\frac{1}{1-\tau}\Theta_2-(1-\upsilon)\Theta_3\end{aligned} \text{ . (4-13-16)}$$

Plugging Eqs.(4-13-12), (4-13-13) into Eq.(4-13-17)' yields:

$$\begin{aligned}\Theta_4&=-\frac{\tau+1-\tau}{1-\tau}\Theta_2-\eta(1-\upsilon)\Theta_3\\&=-\frac{1}{1-\tau}\Theta_2-\eta(1-\upsilon)\Theta_3\end{aligned} \text{ . (4-13-17)}$$

Plugging Eqs.(4-13-12), (4-13-13) into Eq.(4-13-20)' yields:

$$\Theta_2=-\frac{(1-\tau)\Phi}{\tau}-\Theta_3+\frac{\varphi(1-\tau)}{\tau}\Theta_2,$$

which can be rewritten as:

$$\Theta_3=-\frac{(1-\tau)\Phi}{\tau}+\frac{\varphi(1-\tau)-\tau}{\tau}\Theta_2 \text{ . (4-13-18)'}$$

Plugging Eq.(4-13-18)' into Eq.(4-13-16) yields:

$$\begin{aligned}\Theta_5&=-\frac{1}{1-\tau}\Theta_2-(1-\upsilon)\left[-\frac{(1-\tau)\Phi}{\tau}+\frac{\varphi(1-\tau)-\tau}{\tau}\Theta_2\right]\\&=-\frac{1}{1-\tau}\Theta_2+\frac{(1-\upsilon)(1-\tau)\Phi}{\tau}-\frac{(1-\upsilon)[\varphi(1-\tau)-\tau]}{\tau}\Theta_2\\&=\frac{(1-\upsilon)(1-\tau)\Phi}{\tau}-\frac{(1-\upsilon)(1-\tau)[\varphi(1-\tau)-\tau]+\tau}{\tau(1-\tau)}\Theta_2\end{aligned} \text{ . (4-13-21)}$$

Plugging Eq.(4-13-18)' into Eq.(4-13-17) yields:

$$\begin{aligned}\Theta_4&=-\frac{1}{1-\tau}\Theta_2-\eta(1-\upsilon)\left[-\frac{(1-\tau)\Phi}{\tau}+\frac{\varphi(1-\tau)-\tau}{\tau}\Theta_2\right]\\&=\frac{\eta(1-\upsilon)(1-\tau)\Phi}{\tau}-\frac{\eta(1-\upsilon)(1-\tau)[\varphi(1-\tau)-\tau(1-\alpha)]+\tau}{\tau(1-\tau)}\Theta_2\end{aligned} \text{ . (4-13-22)}$$

Plugging Eqs. (4-13-21) and (4-13-22) into Eq.(4-13-18) yields:

$$\Theta_3 = -\frac{2-\upsilon}{1-\upsilon} + \frac{1}{\eta}\left\{\begin{array}{l}\frac{\eta(1-\upsilon)(1-\tau)\Phi}{\tau} \\ -\frac{\eta(1-\upsilon)(1-\tau)\Gamma_1+\tau}{\tau(1-\tau)}\Theta_2\end{array}\right\}$$

$$+\frac{1-\upsilon}{\upsilon\eta}\left\{\frac{(1-\upsilon)(1-\tau)\Phi}{\tau} - \frac{(1-\upsilon)(1-\tau)\Gamma_1+\tau}{\tau(1-\tau)}\Theta_2\right\}$$

$$= -\frac{2-\upsilon}{1-\upsilon} + \frac{(1-\upsilon)(1-\tau)\Phi}{\tau} + \frac{(1-\upsilon)^2(1-\tau)\Phi}{\upsilon\eta\tau}$$

$$-\frac{\eta(1-\upsilon)(1-\tau)\Gamma_1+\tau}{\eta\tau(1-\tau)}\Theta_2$$

$$-\frac{(1-\upsilon)^2(1-\tau)\Gamma_1+(1-\upsilon)\tau}{\upsilon\eta\tau(1-\tau)}\Theta_2$$

$$= \frac{\upsilon\eta(1-\upsilon)^2(1-\tau)\Phi+(1-\upsilon)^2(1-\upsilon)(1-\tau)\Phi-(2-\upsilon)\upsilon\eta\tau}{\upsilon\eta\tau}$$

$$-\left\{\begin{array}{l}\frac{\eta(1-\upsilon)(1-\tau)\Gamma_1+\tau}{\eta\tau(1-\tau)} \\ +\frac{(1-\upsilon)^2(1-\tau)\Gamma_1+(1-\upsilon)\tau}{\upsilon\eta\tau(1-\tau)}\end{array}\right\}\Theta_2$$

$$= \frac{(1-\upsilon)^2(1-\tau)\Phi[\upsilon\eta+(1-\upsilon)]-\upsilon\eta(2-\upsilon)\tau}{\upsilon\eta\tau}$$

$$-\frac{\upsilon\eta(1-\upsilon)(1-\tau)\Gamma_1+\upsilon\tau+(1-\upsilon)^2(1-\tau)\Gamma_1+(1-\upsilon)\tau}{\upsilon\eta\tau(1-\tau)}\Theta_2$$

$$= \frac{(1-\upsilon)^2(1-\tau)\Phi[\upsilon(\eta-1)+1]-\upsilon\eta(2-\upsilon)\tau}{\upsilon\eta\tau}$$

$$-\frac{(1-\upsilon)(1-\tau)\Gamma_1[\upsilon(\eta-1)+1]+(1-\upsilon)\tau+\upsilon\tau}{\upsilon\eta\tau(1-\tau)}\Theta_2$$

$$= \frac{\Phi(1-\upsilon)^2(1-\tau)\Gamma_2-\upsilon\eta(2-\upsilon)\tau}{\upsilon\eta\tau} - \frac{(1-\upsilon)(1-\tau)\Gamma_1\Gamma_2+\tau}{\upsilon\eta\tau(1-\tau)}\Theta_2$$

(4-13-18)’’

With $\Gamma_1 \equiv \varphi(1-\tau)-\tau$, $\Gamma_2 \equiv 1+\upsilon(\eta-1)$ (and $\Gamma_3 \equiv \tau$)

Plugging Eqs.(4-13-18)’ into the previous expression yields:

$$-\frac{(1-\tau)\Phi}{\tau}+\frac{\varphi(1-\tau)-\tau}{\tau}\Theta_2=\frac{\Phi(1-\upsilon)^2(1-\tau)\Gamma_2-\upsilon\eta(2-\upsilon)\tau}{\upsilon\eta\tau}-\frac{(1-\upsilon)(1-\tau)\Gamma_1\Gamma_2+\tau}{\upsilon\eta\tau(1-\tau)}\Theta_2,$$

which can be rewritten as:

$$\frac{\Gamma_1}{\tau}\Theta_2+\frac{(1-\upsilon)(1-\tau)\Gamma_1\Gamma_2+\tau}{\upsilon\eta\tau(1-\tau)}\Theta_2=\frac{\Phi(1-\upsilon)^2(1-\tau)\Gamma_2-\upsilon\eta(2-\upsilon)\tau}{\upsilon\eta\tau}+\frac{(1-\tau)\Phi}{\tau}.$$

Then:

$$\frac{\Gamma_1\upsilon\eta(1-\tau)+(1-\upsilon)(1-\tau)\Gamma_1\Gamma_2+\tau}{\upsilon\eta\tau(1-\tau)}\Theta_2=\frac{\Phi(1-\upsilon)^2(1-\tau)\Gamma_2-\upsilon\eta(2-\upsilon)\tau+\upsilon\eta(1-\tau)\Phi}{\upsilon\eta\tau}$$

Further:

$$\Theta_2=\frac{\upsilon\eta\tau(1-\tau)}{\Gamma_1\upsilon\eta(1-\tau)+(1-\upsilon)(1-\tau)\Gamma_1\Gamma_2+\tau}\frac{\Phi(1-\tau)\left[(1-\upsilon)^2\Gamma_2+\upsilon\eta\right]-\upsilon\eta\tau(2-\upsilon)}{\upsilon\eta\tau}.$$

Finally:

$$\Theta_2=\frac{(1-\tau)\left\{\Phi(1-\tau)\left[(1-\upsilon)^2\Gamma_2+\upsilon\eta\right]-\upsilon\eta\tau(2-\upsilon)\right\}}{\upsilon\left\{\Gamma_1(1-\tau)\left[\upsilon\eta+(1-\upsilon)\Gamma_2\right]+\tau\right\}}. \text{ (4-13-21)}$$

$\Theta 2$ is given by Eq.(4-3-12), $\Theta 3$ is given by Eq.(4-13-18)’, $\Theta 4$ is given by Eq.(4-13-17), $\Theta 5$ is given by Eq.(4-13-16), $\Theta 8$ is given by Eq.(4-13-13) and $\Theta 9$ is given by Eq.(4-13-12).

Then, the solution of Eq.(4-13-1) is given by:

$$
\begin{aligned}
&\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\Phi y_t-\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}\\
&=\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{array}{l}\frac{1}{2}\left(w_t^r\right)^2+\frac{\tau}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{\tau}{2(1-\tau)}x_{H,t}^2-w_t^r a_t\\-c_t w_t^r+c_t a_t+y_t w_t^r-y_t a_t+x_t w_{H,t}^r-x_{H,t}a_t+\frac{\varepsilon_p}{2}\pi_{H,t}^2\end{array}\right\}\\
&+\Theta_3\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{array}{l}\frac{1-\upsilon}{2}c_t^2+\frac{\eta^2(1-\upsilon)}{2}x_{H,t}^2+\frac{\upsilon\eta^2}{2}s_t^2\\-\eta(1-\upsilon)x_{H,t}c_t+\upsilon\eta\sigma_C s_t z_{1,t}^*\end{array}\right\}\\
&+\Theta_4\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\upsilon[\eta(1-\upsilon)-1]}{2}s_t^2\right\}\\
&+\Theta_8\sum_{t=0}^{\infty}\beta^k \mathrm{E}_0\left[\begin{array}{l}\frac{\varphi(2+\varphi)}{2}n_t^2-\frac{1}{2}c_t^2-\frac{1}{2}x_{H,t}^2-\frac{1}{2}\left(w_{t+k}^r\right)^2+c_t n_t+c_{t+k}x_{H,t}\\+c_t w_t^r-n_t x_{H,t}-n_t w_t^r-x_{H,t}w_t^r+\frac{\varepsilon_W(1+\varphi)}{2}\left(\pi_{H,t}^W\right)^2\end{array}\right]\\
&+\Theta_9\sum_{t=0}^{\infty}\beta^t\left[\frac{1}{2}y_t^2+\frac{\varepsilon_p}{2\kappa_p}\pi_{H,t}^2-y_t a_t\right]+\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}\\
&+o\left(\|\xi\|^2\right)
\end{aligned}
\quad , (4\text{-}13\text{-}29)
$$

with:

$$\Upsilon_0\equiv+\Theta_2\kappa^{-1}\overline{\nu}_H+\Theta_8\kappa_W^{-1}\overline{\nu}_H^W .$$

Above results imply that Θ_5 must be disappear.

Now we focus on lines 2 and 2 in Eq.(4-13-29). Plugging Eq.(4-13-13) into lines 2 and 2 in Eq.(4-13-29) yields:

$$
+\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\begin{Bmatrix}\frac{1}{2}\left(w_t^r\right)^2+\frac{\tau}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{\tau}{2(1-\tau)}x_{H,t}^2-w_t^r a_t\\ -c_t w_t^r+c_t a_t+y_t w_t^r-y_t a_t+x_t w_{H,t}^r-x_{H,t}a_t+\frac{\varepsilon_p}{2}\pi_{H,t}^2\end{Bmatrix}
$$

$$
+\Theta_8\sum_{t=0}^{\infty}\beta^k \mathrm{E}_0\begin{bmatrix}\frac{\varphi(2+\varphi)}{2}n_t^2-\frac{1}{2}c_t^2-\frac{1}{2}x_{H,t}^2-\frac{1}{2}\left(w_{t+k}^r\right)^2+c_t n_t+c_{t+k}x_{H,t}\\ +c_t w_t^r-n_t x_{H,t}-n_t w_{H,t}^r-x_{H,t}w_{H,t}^r+\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{bmatrix}
$$

$$
=+\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\begin{Bmatrix}\frac{1}{2}\left(w_t^r\right)^2+\frac{\tau}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{\tau}{2(1-\tau)}x_{H,t}^2-w_t^r a_t\\ -c_t w_t^r+c_t a_t+y_t w_t^r-y_t a_t+x_t w_{H,t}^r-x_{H,t}a_t+\frac{\varepsilon_p}{2}\pi_{H,t}^2\\ +\frac{\varphi(2+\varphi)}{2}n_t^2-\frac{1}{2}c_t^2-\frac{1}{2}x_{H,t}^2-\frac{1}{2}\left(w_{t+k}^r\right)^2+c_t n_t+c_t x_{H,t}\\ +c_t w_t^r-n_t x_{H,t}-n_t w_t^r-x_{H,t}w_t^r+\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{Bmatrix}
$$

$$
=+\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\begin{Bmatrix}\left[\frac{\tau}{2(1-\tau)}-\frac{1}{2}\right]c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\left[\frac{\tau}{2(1-\tau)}+\frac{1}{2}\right]x_{H,t}^2+\frac{\varphi(2+\varphi)}{2}n_t^2\\ +c_t a_t-y_t a_t-x_{H,t}a_t+c_t n_t+c_t x_{H,t}-n_t x_{H,t}\\ +\left[y_t-\left(a_t+n_t\right)\right]w_t^r+\frac{\varepsilon_p}{2}\pi_{H,t}^2+\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{Bmatrix}
$$

$$
=+\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\begin{Bmatrix}\frac{\tau-(1-\tau)}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{1}{2(1-\tau)}x_t^2+\frac{\varphi(2+\varphi)}{2}n_t^2\\ +c_t a_t-y_t a_{H,t}-x_{H,t}a_t+c_t n_t+c_t x_{H,t}-n_t x_{H,t}+\frac{\varepsilon_p}{2}\pi_{H,t}^2\\ +\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{Bmatrix},
$$

where we use Eq.(4-13-33). Plugging the previous expression into Eq.(4-13-29) yields:

$$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\Phi y_t - \frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}$$

$$=\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{array}{l}\frac{\tau-(1-\tau)}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{1}{2(1-\tau)}x_t^2+\frac{\varphi(2+\varphi)}{2}n_t^2\\ +c_t a_t - y_t a_t - x_{H,t}a_t + c_t n_t + c_t x_{H,t} - n_t x_{H,t} + \frac{\varepsilon_p}{2}\pi_{H,t}^2\\ +\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2+\frac{\varphi}{2}y_t^2+\frac{\varphi\varepsilon_p}{2\kappa_p}\pi_{H,t}^2-\varphi y_t a_t\end{array}\right\}$$

$$+\Theta_3\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{1-\upsilon}{2}c_t^2+\frac{\eta^2(1-\upsilon)}{2}x_{H,t}^2+\frac{\upsilon\eta^2}{2}s_t^2-\eta(1-\upsilon)x_{H,t}c_t+\upsilon\eta s_t z_{1,t}^*\right\}$$

$$+\Theta_4\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\upsilon[\eta(1-\upsilon)-1]}{2}s_t^2\right\}$$

$$+\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}$$

$$+o\left(\|\xi\|^2\right)$$

where we use $\Theta_9=\varphi\Theta_8$. Eq.(4-13-12). The 2nd line in the previous expression can be rewritten as:

$$+\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{array}{l}\frac{\tau-(1-\tau)}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{1}{2(1-\tau)}x_t^2+\frac{\varphi(2+\varphi)}{2}n_t^2\\ +c_t a_t - y_t a_{H,t} - x_{H,t}a_t + c_t n_t + c_t x_{H,t} - n_t x_{H,t} + \frac{\varepsilon_p}{2}\pi_{H,t}^2\\ +\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2-\alpha n_t w_t^r+\frac{\varphi}{2}y_t^2+\frac{\varphi\varepsilon_p}{2\kappa_p}\pi_{H,t}^2-(1+\varphi)y_t a_t\end{array}\right\}.$$

$$=+\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{array}{l}\frac{\tau-(1-\tau)}{2(1-\tau)}c_t^2+\frac{\varphi(1-\tau)-\tau}{(1-\tau)2}y_t^2-\frac{1}{2(1-\tau)}x_t^2+\frac{\varphi(2+\varphi)}{2}n_t^2\\ +c_t a_t-(1+\varphi)y_t a_t - x_{H,t}a_t + c_t n_t + c_t x_{H,t} - n_t x_{H,t}\\ +\frac{\varepsilon_p\left[\kappa_p(1-\alpha)+\varphi\right]}{2\kappa_p}\pi_{H,t}^2+\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{array}\right\}$$

Lastly, we have:

, (4-13-34)’

$$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\Phi y_t - \frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}$$

$$= \Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{array}{l}\frac{\tau-(1-\tau)}{2(1-\tau)}c_t^2 + \frac{\Gamma_1}{(1-\tau)2}y_t^2 - \frac{1}{2(1-\tau)}x_t^2 \\ +\frac{\varphi(2+\varphi)}{2}n_t^2 + c_t a_t - (1+\varphi)y_t a_t - x_{H,t}a_t + c_t n_t + c_t x_{H,t} \\ -n_t x_{H,t} + \frac{\varepsilon_p\left(\kappa_p+\varphi\right)}{2\kappa_p}\pi_{H,t}^2 + \frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2 - \alpha n_t w_t^r\end{array}\right\}$$

$$+\Theta_3\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{1-\upsilon}{2}c_t^2 + \frac{\eta^2(1-\upsilon)}{2}x_{H,t}^2 + \frac{\upsilon\eta^2}{2}s_t^2 - \eta(1-\upsilon)x_{H,t}c_t + \upsilon\eta\sigma_C s_t z_{1,t}^*\right\}$$

$$+\Theta_4\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\upsilon[\eta(1-\upsilon)-1]}{2}s_t^2\right\} + \Upsilon_0 + \text{t.i.p.} + o\left(\|\xi\|^2\right)$$

which can be rewritten as:

$$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\Phi y_t - \frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}$$

$$= \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\begin{array}{l}\left[-\frac{\Theta_2}{2(1-\tau)} + \frac{\Theta_3\eta^2(1-\upsilon)\sigma_C}{2}\right]x_{H,t}^2 \\ +\left[\frac{\Theta_2\Gamma_1}{(1-\tau)2}\right]y_t^2 \\ +\left[\frac{\tau-(1-\tau)\Theta_2}{2(1-\tau)} + \frac{\Theta_3(1-\upsilon)}{2}\right]c_t^2 \\ +\left[\frac{\Theta_3\upsilon\eta^2}{2} + \frac{\Theta_4\upsilon[\eta(1-\upsilon)-1]}{2}\right]s_t^2 \\ -[-\Theta_2 + \Theta_3\eta(1-\upsilon)]c_t x_{H,t} + \Theta_2 c_t n_t - \Theta_2 n_t x_{H,t} + \Theta_2 c_t a_t \\ -\Theta_2(1+\varphi)y_t a_t - \Theta_2 x_{H,t}a_t \\ +\Theta_3\upsilon\eta s_t z_{1,t}^* + \frac{\Theta_2\varphi(2+\varphi)}{2}n_t^2 - \alpha\Theta_2 n_t w_t^r \\ +\frac{\Theta_2\varepsilon_p\left[\kappa_p+\varphi\right]}{2\kappa_p}\pi_{H,t}^2 + \frac{\Theta_2\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{array}\right) \quad (4\text{-}13\text{-}35)$$

$$+\mathrm{T}_t + \Upsilon_0 + \text{t.i.p.}$$

$$+o\left(\|\xi\|^2\right)$$

Eq.(3-1-8)’ can be rewritten as:

$$c_t = \frac{1}{(1-\upsilon)\sigma_C} y_t - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)} s_t - \frac{\upsilon}{(1-\upsilon)} z^*_{1,t} \text{. (4-13-30)'}$$

Eq.(3-1-11) can be rewritten as:

$$s_t = \frac{1}{1-\upsilon} c_t - \frac{1}{1-\upsilon} z_t + \frac{1}{1-\upsilon} z^*_{2,t} \text{. (4-13-30)''}$$

Eq.(3-1-26) can be rewritten as:

$$y_t = (1-\alpha) n_t + a_t \text{. (4-13-30)''}$$

Plugging Eq.(4-13-30)’’ into Eq. (4-13-30)’ yields:

$$c_t = \frac{1}{(1-\upsilon)\sigma_C} y_t - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)} \left(\frac{1}{1-\upsilon} c_t - \frac{1}{1-\upsilon} z_t + \frac{1}{1-\upsilon} z^*_{2,t} \right) - \frac{\upsilon}{(1-\upsilon)} z^*_{1,t}$$
$$= \frac{1}{(1-\upsilon)\sigma_C} y_t - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} c_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z_t - \frac{\upsilon}{(1-\upsilon)} z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z^*_{2,t}$$

, which can be rewritten as:

$$\frac{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} c_t = \frac{1}{(1-\upsilon)\sigma_C} y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z_t - \frac{\upsilon}{(1-\upsilon)} z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z^*_{2,t}$$

(4-13-30)’’’

. Finally:

$$c_t = \frac{(1-\upsilon)^2}{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)} \left[\frac{1}{1-\upsilon} y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z_t - \frac{\upsilon}{(1-\upsilon)} z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z^*_{2,t} \right]$$
$$= \frac{(1-\upsilon)}{\left[(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)\right]\sigma_C} y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)} z_t - \frac{\upsilon(1-\upsilon)}{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)} z^*_{1,t}$$
$$- \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)} z^*_{2,t}$$

Plugging eq. (4-13-30)’’ into the previous expression yields:

$$c_t = \frac{(1-\upsilon)}{\gamma_\upsilon} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z^*_{2,t} + \frac{1-\upsilon}{\gamma_\upsilon} a_t \text{. (4-13-30)}$$

with $\gamma_\upsilon \equiv (1-\upsilon)^2 + \eta\upsilon(2-\upsilon)$.

Plugging Eq. (4-13-30) into Eq.(4-13-30)’’ yields:

$$s_t = \frac{1}{1-\upsilon}\left\{\frac{(1-\upsilon)}{\gamma_\upsilon}n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z^*_{2,t} + \frac{(1-\upsilon)}{\gamma_\upsilon\sigma_C}a_t\right\}$$
$$-\frac{1}{1-\upsilon}z_t + \frac{1}{1-\upsilon}z^*_{2,t}$$
$$= \frac{1}{\gamma_\upsilon}n_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon}z_t - \frac{\upsilon}{\gamma_\upsilon}z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon}z^*_{2,t} + \frac{1}{\gamma_\upsilon}a_t$$

. (4-13-32)

Plugging Eq.(4-13-32) into Eq.(4-8-3) yields:

$$x_{H,t} = -\upsilon\left[\frac{1}{\gamma_\upsilon}n_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon}z_t - \frac{\upsilon}{\gamma_\upsilon}z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon}z^*_{2,t} + \frac{1}{\gamma_\upsilon}a_t\right]$$
$$= -\frac{\upsilon}{\gamma_\upsilon}n_t - \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon}z_t + \frac{\upsilon^2}{\gamma_\upsilon}z^*_{1,t} + \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon}z^*_{2,t} - \frac{\upsilon}{\gamma_\upsilon}a_t$$

. (4-13-31)

Let plug Eqs.(4-13-30) to (4-13-32) and (4-1-30)” into the second-order and cross terms in Eq.(4-13-35). Then we have:

$$x^2_{H,t} = \left\{\begin{aligned}&-\frac{\upsilon}{\gamma_\upsilon}n_t - \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon}z_t + \frac{\upsilon^2}{\gamma_\upsilon}z^*_{1,t} + \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon}z^*_{2,t}\\&-\frac{\upsilon}{\gamma_\upsilon\sigma_C}a_t\end{aligned}\right\}^2$$
$$= \frac{\upsilon^2}{(\gamma_\upsilon)^2}n_t^2 + \frac{2\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon^2}n_t z_t - \frac{2\upsilon^3}{\gamma_\upsilon^2}n_t z^*_{1,t}$$
$$-\frac{2\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon^2}n_t z^*_{2,t} + \frac{2\upsilon^2}{(\gamma_\upsilon)^2}n_t a_t$$
$$+\text{s.o.t.i.p}$$

, (4-13-36)

$$y_t^2 = (n_t + a_t)^2$$
$$= n_t^2 + 2n_{H,t}a_{H,t} + \text{s.o.t.i.p}$$

, (4-13-37)

$$c_t^2 = \left[\frac{(1-\upsilon)}{\gamma_\upsilon}n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z^*_{2,t} + \frac{1-\upsilon}{\gamma_\upsilon}a_t\right]^2$$
$$= \frac{(1-\upsilon)^2}{(\gamma_\upsilon)^2}n_t^2 + \frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)}{\gamma_\upsilon^2}n_t z_t - \frac{2\upsilon(1-\upsilon)^2}{\gamma_\upsilon^2}n_t z^*_{1,t}$$
$$-\frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)}{\gamma_\upsilon^2}n_t z^*_{2,t} + \frac{2(1-\upsilon)^2}{\gamma_\upsilon^2}n_t a_t + \text{s.o.t.i.p}$$

, (4-13-38)

with $\gamma_C \equiv (1-\upsilon)^2$ and $\gamma_D \equiv \eta(2-\upsilon)(1-\upsilon)$,

$$
\begin{aligned}
s_t^{\,2} &= \left[\frac{1}{\gamma_\upsilon}n_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon}z_t - \frac{\upsilon}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon}z_{2,t}^* + \frac{1}{\gamma_\upsilon}a_t\right]^2 \\
&= \frac{1}{(\gamma_\upsilon)^2}n_t^2 + \frac{2(1-\alpha)[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2}n_t z_t - \frac{\upsilon}{\gamma_\upsilon^2}n_t z_{1,t}^* \\
&\quad -\frac{2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2}n_t z_{2,t}^* + \frac{1}{\gamma_\upsilon^2}n_t a_t + \text{s.o.t.i.p}
\end{aligned}
$$

$$
\begin{aligned}
c_t x_{H,t} &= \begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon}n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t \\ -\frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_{2,t}^* \\ +\frac{(1-\upsilon)}{\gamma_\upsilon}a_t \end{bmatrix}\begin{bmatrix} -\frac{\upsilon}{\gamma_\upsilon}n_t \\ -\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon}z_t + \frac{\upsilon^2}{\gamma_\upsilon}z_{1,t}^* \\ +\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon}z_{2,t}^* - \frac{\upsilon}{\gamma_\upsilon}a_t \end{bmatrix} \\
&= -\frac{\upsilon(1-\upsilon)}{\gamma_\upsilon^2}n_t^2 - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{\gamma_\upsilon^2}n_t z_t + \frac{\upsilon^2(1-\upsilon)}{\gamma_\upsilon^2}n_t z_{1,t}^* \\
&\quad + \frac{\upsilon(1-\upsilon)[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2}n_t z_{2,t}^* - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon^2}n_t a_t - \frac{\upsilon^2\eta(2-\upsilon)}{\gamma_\upsilon^2}n_t z_t \\
&\quad + \frac{\upsilon^2(1-\upsilon)}{\gamma_\upsilon^2}n_t z_{1,t}^* + \frac{\upsilon^2\eta(2-\upsilon)}{\gamma_\upsilon^2}n_t z_{2,t}^* - \frac{\upsilon(1-\upsilon)}{(\gamma_\upsilon)^2}n_t a_t + \text{s.o.t.i.p} \\
&= -\frac{\upsilon(1-\upsilon)}{(\gamma_\upsilon)^2}n_t^2 - \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]}{\gamma_\upsilon^2}n_t z_t + \frac{\upsilon^2 2(1-\upsilon)}{\gamma_\upsilon^2}n_t z_{1,t}^* \\
&\quad + \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]}{\gamma_\upsilon^2}n_t z_{2,t}^* - \frac{\upsilon 2(1-\upsilon)}{(\gamma_\upsilon)^2}n_t a_t + \text{s.o.t.i.p}
\end{aligned}
\quad , (4\text{-}13\text{-}39)
$$

$$
\begin{aligned}
c_t n_t &= \begin{bmatrix} \frac{(1-\upsilon)(1-\alpha)}{\gamma_\upsilon}n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_{2,t}^* \\ +\frac{(1-\upsilon)}{\gamma_\upsilon}a_t \end{bmatrix} n_t \\
&= \frac{(1-\upsilon)(1-\alpha)}{\gamma_\upsilon}n_t^2 + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}n_t z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}n_t z_{1,t}^* \\
&\quad - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}n_t z_{2,t}^* + \frac{(1-\upsilon)}{\gamma_\upsilon}n_t a_t
\end{aligned}
\quad , (4\text{-}13\text{-}42)
$$

$$n_t x_{H,t} = n_t \left\{ \begin{array}{l} -\frac{\upsilon(1-\alpha)}{\gamma_\upsilon} n_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z_{1,t}^* \\ + \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* - \frac{\upsilon}{\gamma_\upsilon} a_t \end{array} \right\}$$

$$= -\frac{\upsilon(1-\alpha)}{\gamma_\upsilon} n_t^2 - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} n_t z_t + \frac{\upsilon^2}{\gamma_\upsilon} n_t z_{1,t}^* \quad , (4\text{-}13\text{-}43)$$

$$+ \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} n_t z_{2,t}^* - \frac{\upsilon}{\gamma_\upsilon} n_t a_t$$

$$c_t a_t = \left[\frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* + \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} a_t \right] a_t \quad , (4\text{-}13\text{-}51)$$

$$= \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t a_t + \text{t.i.p}$$

$$y_t a_t = [n_t + a_t] a_t \quad , (4\text{-}13\text{-}49)'$$

$$= n_t a_t + \text{s.o.t.i.p}$$

$$x_t a_t = \left\{ \begin{array}{l} -\frac{\upsilon}{\gamma_\upsilon} n_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z_{1,t}^* + \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* \\ -\frac{\upsilon}{\gamma_\upsilon} a_t \end{array} \right\} a_t \quad , (4\text{-}13\text{-}52)$$

$$= -\frac{\upsilon}{\gamma_\upsilon} n_t a_t + \text{s.o.t.i.p}$$

$$s_t z_{1,t}^* = \left[\begin{array}{l} \frac{1}{\gamma_\upsilon} n_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* \\ + \frac{1}{\gamma_\upsilon \sigma_C} a_t \end{array} \right] z_{1,t}^* \quad , (4\text{-}13\text{-}53\text{-}e)$$

$$= \frac{1-\alpha}{\gamma_\upsilon} n_t z_{1,t}^* + \text{s.o.t.i.p}$$

where we use Eqs.(4-11-6) and (4-11-7) to derive Eq.(4-13-36).

Now we calculate coefficients of in Eq.(4-13-35). These are given by:

$$\left[-\frac{1}{2(1-\tau)}\Theta_2+\frac{\eta^2(1-\upsilon)}{2}\Theta_3\right]x_t^2=\frac{-\Theta_2+(1-\tau)\eta^2(1-\upsilon)\Theta_3}{2(1-\tau)}x_t^2,$$

$$=\frac{(1-\tau)\left[\eta^2(1-\upsilon)\Theta_3\right]-\Theta_2}{2(1-\tau)}x_t^2$$

$$\left[\frac{\Theta_2\Gamma_1}{(1-\tau)2}\right]y_t^2=\frac{(1-\tau)\Theta_1+\Theta_2(1-\beta)\sigma_B\Gamma_1}{2(1-\beta)(1-\tau)\sigma_B}y_t^2$$

$$\left[\frac{\tau-(1-\tau)\Theta_2}{2(1-\tau)}+\frac{\Theta_3(1-\upsilon)}{2}\right]c_t^2=\frac{\left[\tau-(1-\tau)\right]\Theta_2+(1-\tau)(1-\upsilon)\Theta_3}{2(1-\tau)}c_t^2$$

$$=\frac{(2\tau-1)\Theta_2+(1-\tau)(1-\upsilon)\Theta_3}{2(1-\tau)}c_t^2$$

$$=\frac{(1-\tau)\Theta_3(1-\upsilon)-(1-2\tau)\Theta_2}{2(1-\tau)}c_t^2$$

$$\left[\frac{\Theta_3\upsilon\eta^2}{2}+\frac{\Theta_4\upsilon\left[\eta(1-\upsilon)-1\right]}{2}\right]s_t^2=\frac{\upsilon\left\{\Theta_3\eta^2+\Theta_4\left[\eta(1-\upsilon)-1\right]\right\}}{2}s_t^2$$

, and we define as follows:

$x_{H,t}^2$: $\Gamma_{11}\equiv\dfrac{(1-\tau)\eta^2(1-\upsilon)\Theta_3-\Theta_2}{2(1-\tau)}$, (4-13-54)

y_t^2: $\Gamma_{12}\equiv\dfrac{\Theta_2\Gamma_1}{2(1-\tau)}$, (4-13-55)

c_t^2: $\Gamma_{13}\equiv\dfrac{(1-\tau)\Theta_3(1-\upsilon)-(1-2\tau)\Theta_2}{2(1-\tau)}$, (4-13-56)

s_t^2: $\Gamma_{14}\equiv\dfrac{\Theta_3\eta^2+\Theta_4\left[\eta(1-\upsilon)-1\right]}{2}$, (4-13-57)

$c_t x_{H,t}$: $\Gamma_{15}\equiv-\Theta_2+\Theta_3\eta(1-\upsilon)$, (4-13-58).

By Plugging Eqs.(4-13-36) to (4-13-39) and (4-13-54) to (4-14-58) into lines 3 to 7 in Eq. (4-14-35), we have:

$$
\left[-\frac{\Theta_2}{2(1-\tau)}+\frac{\Theta_3\eta^2(1-\upsilon)\sigma_C}{2}\right]x_{H,t}^2+\left[\frac{\Theta_2\left[\varphi(1-\tau)-\tau(1-\alpha)\right]}{(1-\alpha)(1-\tau)2}\right]y_t^2
$$

$$
+\left[\frac{\tau-(1-\tau)\Theta_2}{2(1-\tau)}+\frac{\Theta_3(1-\upsilon)\sigma_C}{2}\right]c_t^2+\left[\frac{\Theta_3\upsilon\eta^2\sigma_C}{2}+\frac{\Theta_4\upsilon\left[\eta(1-\upsilon)-1\right]}{2}\right]s_t^2
$$

$$
-\left[-\Theta_2+\Theta_3\eta(1-\upsilon)\sigma_C\right]c_t x_{H,t}
$$

$$
=\Gamma_{11}\left\{\begin{array}{l}\frac{\upsilon^2}{(\gamma_\upsilon)^2}n_t^2+\frac{2\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon^2}n_t z_t-\frac{2\upsilon^3}{\gamma_\upsilon^2}n_t z_{1,t}^*\\ -\frac{2\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon^2}n_t z_{2,t}^*+\frac{2\upsilon^2}{\gamma_\upsilon^2}n_t a_t\end{array}\right\}
$$

$$
+\Gamma_{12}\left(n_t^2+2n_t a_{H,t}\right)
$$

$$
+\Gamma_{13}\left[\begin{array}{l}\frac{(1-\upsilon)^2}{\gamma_\upsilon^2}n_t^2+\frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)}{\gamma_\upsilon^2}n_t z_t-\frac{2\upsilon(1-\upsilon)^2}{\gamma_\upsilon^2}n_t z_{1,t}^*\\ -\frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)}{\gamma_\upsilon^2}n_t z_{2,t}^*+\frac{2(1-\upsilon)^2}{(\gamma_\upsilon)^2}n_t a_t\end{array}\right]
$$

$$
+\Gamma_{14}\left\{\begin{array}{l}\frac{1}{(\gamma_\upsilon)^2}n_t^2+\frac{2(1-\alpha)\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon^2}n_t z_t-\frac{\upsilon}{\gamma_\upsilon^2}n_t z_{1,t}^*\\ -\frac{2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon^2}n_t z_{2,t}^*+\frac{1}{\gamma_\upsilon^2}n_t a_t\end{array}\right\}
$$

$$
-\Gamma_{15}\left\{\begin{array}{l}-\frac{\upsilon(1-\upsilon)}{(\gamma_\upsilon)^2}n_t^2-\frac{\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]}{\gamma_\upsilon^2}n_t z_t+\frac{\upsilon^2 2(1-\upsilon)}{\gamma_\upsilon^2}n_t z_{1,t}^*\\ +\frac{\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]}{\gamma_\upsilon^2}n_t z_{2,t}^*-\frac{\upsilon 2(1-\upsilon)}{(\gamma_\upsilon)^2}n_t a_t\end{array}\right\}
$$

$$
\begin{aligned}
&= \frac{\upsilon^2\Gamma_{11}}{(\gamma_\upsilon)^2} n_t^2 + \frac{2\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{11}}{(1-\upsilon)\gamma_\upsilon^2} n_t z_t - \frac{2\upsilon^3\Gamma_{11}}{\gamma_\upsilon^2} n_t z_{1,t}^* \\
&- \frac{2\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{11}}{(1-\upsilon)\gamma_\upsilon^2} n_t z_{2,t}^* + \frac{2\upsilon^2\Gamma_{11}}{(\gamma_\upsilon\sigma_C)^2} n_t a_t + \Gamma_{12} n_t^2 + 2\Gamma_{12} n_t a_{H,t} \\
&+ \frac{(1-\upsilon)^2\Gamma_{13}}{(\gamma_\upsilon)^2} n_t^2 + \frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}}{\gamma_\upsilon^2} n_t z_t - \frac{2\upsilon(1-\upsilon)^2\Gamma_{13}}{\gamma_\upsilon^2} n_t z_{1,t}^* \\
&- \frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}}{\gamma_\upsilon^2} n_t z_{2,t}^* + \frac{2(1-\upsilon)^2\Gamma_{13}}{(\gamma_\upsilon)^2} n_t a_t + \frac{\Gamma_{14}}{(\gamma_\upsilon)^2} n_t^2 \\
&+ \frac{2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{14}}{(1-\upsilon)\gamma_\upsilon^2} n_t z_t - \frac{\upsilon\Gamma_{14}}{\gamma_\upsilon^2} n_t z_{1,t}^* - \frac{2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{14}}{(1-\upsilon)\gamma_\upsilon^2} n_t z_{2,t}^* + \frac{\Gamma_{14}}{(\gamma_\upsilon)^2} n_t a_t \\
&\frac{\upsilon(1-\upsilon)\Gamma_{15}}{(\gamma_\upsilon)^2} n_t^2 + \frac{\upsilon(1-\alpha)\left[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{15}}{\gamma_\upsilon^2} n_t z_t \\
&- \frac{\upsilon^2 2(1-\upsilon)\Gamma_{15}}{\gamma_\upsilon^2} n_t z_{1,t}^* - \frac{\upsilon(1-\alpha)\left[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{15}}{\gamma_\upsilon^2} n_t z_{2,t}^* \\
&+ \frac{\upsilon 2(1-\upsilon)\Gamma_{15}}{(\gamma_\upsilon\sigma_C)^2} n_t a_t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\upsilon^2\Gamma_{11}}{(\gamma_\upsilon)^2} n_t^2 + \Gamma_{12} n_t^2 + \frac{(1-\upsilon)^2\Gamma_{13}}{(\gamma_\upsilon)^2} n_t^2 + \frac{\Gamma_{14}}{(\gamma_\upsilon)^2} n_t^2 + \frac{\upsilon(1-\upsilon)\Gamma_{15}}{(\gamma_\upsilon)^2} n_t^2 \\
&+ \frac{2\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{11}}{(1-\upsilon)\gamma_\upsilon^2} n_t z_t + \frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}}{\gamma_\upsilon^2} n_t z_t \\
&+ \frac{2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{14}}{(1-\upsilon)\gamma_\upsilon^2} n_t z_t + \frac{\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{15}}{\gamma_\upsilon^2} n_t z_t \\
&- \frac{2\upsilon^3\Gamma_{11}}{\gamma_\upsilon^2} n_t z_{1,t}^* - \frac{2\upsilon(1-\upsilon)^2\Gamma_{13}}{\gamma_\upsilon^2} n_t z_{1,t}^* - \frac{\upsilon\Gamma_{14}}{\gamma_\upsilon^2} n_t z_{1,t}^* - \frac{\upsilon^2 2(1-\upsilon)\Gamma_{15}}{\gamma_\upsilon^2} n_t z_{1,t}^* \\
&- \frac{2\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{11}}{(1-\upsilon)\gamma_\upsilon^2} n_t z_{2,t}^* - \frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}}{\gamma_\upsilon^2\sigma_C} n_t z_{2,t}^* \\
&- \frac{2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{14}}{(1-\upsilon)\gamma_\upsilon^2} n_t z_{2,t}^* - \frac{\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{15}}{\gamma_\upsilon^2} n_t z_{2,t}^* \\
&+ \frac{2\upsilon^2\Gamma_{11}}{(\gamma_\upsilon)^2} n_t a_t + 2\Gamma_{12} n_t a_t + \frac{2(1-\upsilon)^2\Gamma_{13}}{(\gamma_\upsilon)^2} n_t a_t + \frac{\Gamma_{14}}{(\gamma_\upsilon)^2} n_t a_t + \frac{\upsilon 2(1-\upsilon)\Gamma_{15}}{(\gamma_\upsilon)^2} n_t a_t
\end{aligned}
$$

$$=\frac{1}{(\gamma_v)^2}\left[v^2\Gamma_{11}+(\gamma_v\sigma_C)^2\Gamma_{12}+(1-v)^2\Gamma_{13}+\Gamma_{14}+v(1-v)\Gamma_{15}\right]n_t^2$$

$$-\frac{2}{\gamma_v^2}\left\{\begin{array}{l}-\frac{v^2[\eta v(2-v)-\gamma_v]\Gamma_{11}}{1-v}-v\eta(2-v)(1-v)\Gamma_{13}\\-\frac{[\eta v(2-v)-\gamma_v]\Gamma_{14}}{1-v}-\frac{v[2v\eta(2-v)-\gamma_v]\Gamma_{15}}{2}\end{array}\right\}n_t z_t$$

$$-\frac{2v}{\gamma_v^2}\left[v^2\Gamma_{11}+(1-v)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}+v(1-v)\Gamma_{15}\right]n_t z_{1,t}^*$$

$$-\frac{2}{\gamma_v^2}\left[\begin{array}{l}\frac{v^2[\eta v(2-v)-\gamma_v]\Gamma_{11}}{(1-v)}+v\eta(2-v)(1-v)\Gamma_{13}\\+\frac{[\eta v(2-v)-\gamma_v]\Gamma_{14}}{(1-v)}+\frac{v[2v\eta(2-v)-\gamma_v]\Gamma_{15}}{2}\end{array}\right]n_t z_{2,t}^*$$

$$-\frac{2}{(\gamma_v)^2}\left[-v^2\Gamma_{11}-(\gamma_v)^2\Gamma_{12}-(1-v)^2\Gamma_{13}-\frac{\Gamma_{14}}{2}-v(1-v)\Gamma_{15}\right]n_t a_t$$

$$=\frac{1}{(\gamma_v)^2}\left\{\left[v^2\Gamma_{11}+(1-v)^2\Gamma_{13}+\Gamma_{14}\right]+(\gamma_v)^2\Gamma_{12}+v(1-v)\Gamma_{15}\right\}n_t^2$$

$$-\frac{2}{\gamma_v^2}\left\{\begin{array}{l}-\frac{v^2[\eta v(2-v)-\gamma_v]\Gamma_{11}}{1-v}-v\eta(2-v)(1-v)\Gamma_{13}\\-\frac{[\eta v(2-v)-\gamma_v]\Gamma_{14}}{1-v}-\frac{v[2v\eta(2-v)-\gamma_v]\Gamma_{15}}{2}\end{array}\right\}n_t z_t$$

$$-\frac{2v}{\gamma_v^2}\left[\begin{array}{l}v^2\Gamma_{11}+(1-v)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}\\+v(1-v)\Gamma_{15}\end{array}\right]n_t z_{1,t}^*$$

$$-\frac{2}{\gamma_v^2}\left[\begin{array}{l}\frac{v^2[\eta v(2-v)-\gamma_v]\Gamma_{11}}{(1-v)}+v\eta(2-v)(1-v)\Gamma_{13}\\+\frac{[\eta v(2-v)-\gamma_v]\Gamma_{14}}{(1-v)}+\frac{v[2v\eta(2-v)-\gamma_v]\Gamma_{15}}{2}\end{array}\right]n_t z_{2,t}^*$$

$$-\frac{2}{(\gamma_v)^2}\left[\begin{array}{l}-v^2\Gamma_{11}-(\gamma_v)^2\Gamma_{12}-(1-v)^2\Gamma_{13}\\-\frac{\Gamma_{14}}{2}-v(1-v)\Gamma_{15}\end{array}\right]n_t a_t$$

$$=\frac{1}{(\gamma_\upsilon\sigma_C)^2}\left\{\left[\upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\Gamma_{14}\right]+(\gamma_\upsilon\sigma_C)^2\Gamma_{12}+\upsilon(1-\upsilon)\Gamma_{15}\right\}n_t^2$$

$$-\frac{2}{\gamma_\upsilon^2\sigma_C}\left[\begin{array}{l}-\frac{\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{11}}{1-\upsilon}-\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}\\-\frac{[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{14}}{1-\upsilon}-\frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]\Gamma_{15}}{2}\end{array}\right]n_tz_t$$

$$-\frac{2\upsilon}{\gamma_\upsilon^2\sigma_C}\left[\upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}+\upsilon(1-\upsilon)\Gamma_{15}\right]n_tz_{1,t}^*$$

$$-\frac{2}{\gamma_\upsilon^2\sigma_C}\left[\begin{array}{l}\frac{\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{11}}{(1-\upsilon)}+\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}\\+\frac{[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{14}}{(1-\upsilon)}+\frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]\Gamma_{15}}{2}\end{array}\right]n_tz_{2,t}^*$$

$$-\frac{2}{(\gamma_\upsilon\sigma_C)^2}\left[-\upsilon^2\Gamma_{11}-(\gamma_\upsilon\sigma_C)^2\Gamma_{12}-(1-\upsilon)^2\Gamma_{13}-\frac{\Gamma_{14}}{2}-\upsilon(1-\upsilon)\Gamma_{15}\right]n_ta_t$$

(4-13-35a)

By Plugging Eqs. (4-13-40) to (4-13-53e) into lines 8 to 12 in Eq. (4-14-35), we have:

$$\Theta_2c_tn_t-\Theta_2n_tx_{H,t}+\Theta_2c_ta_t-\Theta_2[1+\varphi]y_ta_t-\Theta_2x_{H,t}a_t+\Theta_3\upsilon\eta s_tz_{1,t}^*$$

$$=\Theta_2\left\{\begin{array}{l}\frac{1-\upsilon}{\gamma_\upsilon}n_t^2+\frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}n_tz_t-\frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}n_tz_{1,t}^*-\frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}n_tz_{2,t}^*\\+\frac{(1-\upsilon)}{\gamma_\upsilon\sigma_C}n_ta_t\end{array}\right\}$$

$$-\Theta_2\left\{\begin{array}{l}-\frac{\upsilon}{\gamma_\upsilon}n_t^2-\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon}n_tz_t+\frac{\upsilon^2}{\gamma_\upsilon}n_tz_{1,t}^*\\+\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon}n_tz_{2,t}^*-\frac{\upsilon}{\gamma_\upsilon}n_ta_t\end{array}\right\}$$

$$+\frac{\Theta_2(1-\upsilon)}{\gamma_\upsilon}n_ta_t-\Theta_2[1+\varphi]n_ta_t-\Theta_2\left[-\frac{\upsilon}{\gamma_\upsilon}n_ta_t\right]+\frac{\Theta_3\upsilon\eta}{\gamma_\upsilon}n_tz_{1,t}^*$$

$$
\begin{aligned}
&= \frac{(1-\upsilon)\Theta_2}{\gamma_\upsilon} n_t^2 + \frac{\eta\upsilon(2-\upsilon)\Theta_2}{\gamma_\upsilon} n_t z_t - \frac{\upsilon(1-\upsilon)\Theta_2}{\gamma_\upsilon} n_t z_{1,t}^* \\
&- \frac{\eta\upsilon(2-\upsilon)\Theta_2}{\gamma_\upsilon} n_t z_{2,t}^* - + \frac{(1-\upsilon)\Theta_2}{\gamma_\upsilon} n_t a_t \\
&+ \frac{\upsilon(1-\alpha)\Theta_2}{\gamma_\upsilon} n_t^2 + \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{(1-\upsilon)\gamma_\upsilon} n_t z_t - \frac{\upsilon^2\Theta_2}{\gamma_\upsilon} n_t z_{1,t}^* \\
&- \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{(1-\upsilon)\gamma_\upsilon} n_t z_{2,t}^* + \frac{\upsilon\Theta_2}{\gamma_\upsilon\sigma_C} n_t a_t \\
&+ \frac{\Theta_2(1-\upsilon)}{\gamma_\upsilon} n_t a_t - \Theta_2[1+\varphi] n_t a_t + \frac{\upsilon\Theta_2}{\gamma_\upsilon} n_t a_t + \frac{\Theta_3\upsilon\eta}{\gamma_\upsilon} n_t z_{1,t}^*
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Theta_2}{\gamma_\upsilon} n_t^2 \\
&- \frac{2}{\gamma_\upsilon}\left[-\frac{\eta\upsilon(2-\upsilon)\Theta_2}{2} - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{2(1-\upsilon)} \right] n_t z_t \\
&- \frac{2\upsilon}{\gamma_\upsilon}\left[\frac{\Theta_2}{2} - \frac{\eta\Theta_3}{2} \right] n_t z_{1,t}^* \\
&- \frac{2\upsilon}{\gamma_\upsilon}\left[\frac{\eta(2-\upsilon)\Theta_2}{2} + \frac{[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{(1-\upsilon)2} \right] n_t z_{2,t}^* \\
&- \frac{2}{\gamma_\upsilon}\left[\begin{array}{l} -\dfrac{(1-\upsilon)\Theta_2}{2} - \dfrac{\upsilon\Theta_2}{2} - \dfrac{\Theta_2(1-\upsilon)}{2} \\ + \dfrac{\Theta_2\gamma_\upsilon\sigma_C[1+\varphi]}{2} - \dfrac{\upsilon\Theta_2}{2} \end{array} \right] n_t a_t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Theta_2}{\gamma_\upsilon} n_t^2 \\
&- \frac{2}{\gamma_\upsilon}\left[-\frac{\eta\upsilon(2-\upsilon)\Theta_2}{2} - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{2(1-\upsilon)} \right] n_t z_t \\
&- \frac{2\upsilon}{\gamma_\upsilon^2}\left[\frac{\gamma_\upsilon\Theta_2}{2} - \frac{\eta\gamma_\upsilon\Theta_3}{2} \right] n_t z_{1,t}^* \\
&- \frac{2\upsilon}{\gamma_\upsilon}\left[\frac{\eta(2-\upsilon)\Theta_2}{2} + \frac{[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{(1-\upsilon)2} \right] n_t z_{2,t}^* \\
&- \frac{2}{\gamma_\upsilon}\left[-\frac{\Theta_2}{2} - \frac{\Theta_2}{2} + \frac{\Theta_2\gamma_\upsilon[1+\varphi]}{2} \right] n_t a_t
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\Theta_2}{\gamma_v}n_t^2 \\
&-\frac{2}{\gamma_v}\left[-\frac{\eta\upsilon(2-\upsilon)\Theta_2}{2}-\frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_v\right]\Theta_2}{2(1-\upsilon)}\right]n_t z_t \\
&-\frac{2\upsilon}{\gamma_v}\left[-\frac{\eta\Theta_3-\Theta_2}{2}\right]n_t z_{1,t}^* \\
&-\frac{2\upsilon}{\gamma_v}\left[+\frac{\eta(2-\upsilon)\Theta_2}{2}+\frac{\left[\eta\upsilon(2-\upsilon)-\gamma_v\right]\Theta_2}{(1-\upsilon)2}\right]n_t z_{2,t}^* \\
&-\frac{2}{\gamma_v}\left[-\frac{2\Theta_2}{2}+\frac{\Theta_2\gamma_v[1+\varphi]}{2}\right]n_t a_t
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{\Theta_2}{\gamma_v}n_t^2 \\
&-\frac{2}{\gamma_v}\left[-\frac{\upsilon\Theta_2\eta(2-\upsilon)}{2}+\frac{\upsilon\Theta_2\left[\gamma_v-\eta\upsilon(2-\upsilon)\right]}{2(1-\upsilon)}\right]n_t z_t \\
&-\frac{2\upsilon}{\gamma_v}\frac{\left[\Theta_2-\eta\Theta_3\right]}{2}n_t z_{1,t}^* \\
&-\frac{2\upsilon}{\gamma_v}\left[\frac{\eta(2-\upsilon)\Theta_2}{2}+\frac{\left[\eta\upsilon(2-\upsilon)-\gamma_v\right]\Theta_2}{(1-\upsilon)2}\right]n_t z_{2,t}^* \\
&-\frac{2}{\gamma_v}\frac{\Theta_2\left\{2+\gamma_v[1+\varphi]\right\}}{2}n_t a_t
\end{aligned}
\qquad . \text{(4-13-35b)}
$$

Combinig Eqs.(4-13-35a) and (4-13-35b) yields:

$$\frac{1}{(\gamma_v)^2}\begin{pmatrix}\left\{\left[\upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\Gamma_{14}\right]+\Theta_2\gamma_v\right\}\\+(\gamma_v\sigma_C)^2\Gamma_{12}+\upsilon(1-\upsilon)\Gamma_{15}\end{pmatrix}n_t^2$$

$$-\frac{2}{\gamma_v^2}\left\{\begin{array}{l}\dfrac{\begin{array}{l}\left[\eta\upsilon(2-\upsilon)-\gamma_v\right]\left\{\left[-2\left(\upsilon^2\Gamma_{11}+\Gamma_{14}\right)\right]-\upsilon\gamma_v\Theta_2\right\}\\-\upsilon\gamma_v\Theta_2\eta(1-\upsilon)(2-\upsilon)\end{array}}{2(1-\upsilon)}\\+\dfrac{\left[-\upsilon\eta(2-\upsilon)(1-\upsilon)2\Gamma_{13}-\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_v\right]\Gamma_{15}\right]}{2}\end{array}\right\}n_tz_t$$

$$-\frac{2\upsilon}{\gamma_v^2}\left\{\left[\upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}+\upsilon(1-\upsilon)\Gamma_{15}\right]-\frac{\gamma_v\left[\eta(1-\alpha)\Theta_3-\Theta_2\right]}{2}\right\}n_tz_{1,t}^*$$

$$-\frac{2}{\gamma_v^2}\left\{\begin{array}{l}\left[\begin{array}{l}\dfrac{\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_v\right]\Gamma_{11}}{(1-\upsilon)}+\dfrac{\upsilon\eta(2-\upsilon)(1-\upsilon)^2\Gamma_{13}}{(1-\upsilon)}\\+\dfrac{\left[\eta\upsilon(2-\upsilon)-\gamma_v\right]\Gamma_{14}}{(1-\upsilon)}+\dfrac{\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_v\right]\Gamma_{15}}{2}\end{array}\right]\\+\dfrac{\gamma_v\Theta_2\left\{\eta(2-\upsilon)(1-\upsilon)+\eta(2-\upsilon)\upsilon-\gamma_v\right\}}{(1-\upsilon)2}\end{array}\right\}n_tz_{2,t}^*$$

$$-\frac{2}{(\gamma_v)^2}\left\{\begin{array}{l}\left[\left[-\upsilon^2\Gamma_{11}-(\gamma_v\sigma_C)^2\Gamma_{12}-(1-\upsilon)^2\Gamma_{13}-\dfrac{\Gamma_{14}}{2}-\upsilon(1-\upsilon)\Gamma_{15}\right]\right]\\+\dfrac{\Theta_2\gamma_v\left\{2+\gamma_v(1+\varphi)\right\}}{2}\end{array}\right\}n_ta_t$$

$$=\frac{1}{(\gamma_\upsilon)^2}\begin{Bmatrix}\left\{\left[\upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\Gamma_{14}\right]+\Theta_2\gamma_\upsilon\right\}\\+(\gamma_\upsilon)^2\Gamma_{12}+\upsilon(1-\upsilon)\Gamma_{15}\end{Bmatrix}n_t^2$$

$$-\frac{2}{\gamma_\upsilon^2}\left\{\begin{matrix}+\left\{\begin{matrix}\left[\frac{\left[-\upsilon\eta(2-\upsilon)(1-\upsilon)2\Gamma_{13}-\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{15}\right]}{2}\right]\\+\frac{\left[\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]\left[-2\left(\upsilon^2\Gamma_{11}+\Gamma_{14}\right)\right]}{2(1-\upsilon)}\end{matrix}\right\}\\-\frac{\left[\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]\upsilon\gamma_\upsilon\sigma_C\Theta_2+\upsilon\sigma_C\gamma_\upsilon\Theta_2\eta(1-\upsilon)(2-\upsilon)}{2(1-\upsilon)}\end{matrix}\right\}n_t z_t$$

$$-\frac{2\upsilon}{\gamma_\upsilon^2}\left\{\begin{matrix}\left[\upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}+\upsilon(1-\upsilon)\Gamma_{15}\right]\\-\frac{\gamma_\upsilon\left[\eta\Theta_3-\Theta_2\right]}{2}\end{matrix}\right\}n_t z_{1,t}^*$$

$$-\frac{2}{\gamma_\upsilon^2}\left\{\begin{matrix}\left[\begin{matrix}\frac{\upsilon^2\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{11}+(1-\upsilon)^2\upsilon\eta(2-\upsilon)\Gamma_{13}}{(1-\upsilon)}\\+\frac{\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{14}}{(1-\upsilon)}+\frac{\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{15}}{2}\end{matrix}\right]\\+\frac{\gamma_\upsilon\sigma_C\Theta_2\left\{\eta(2-\upsilon)(1-\upsilon)+\eta(2-\upsilon)\upsilon-\gamma_\upsilon\right\}}{(1-\upsilon)2}\end{matrix}\right\}n_t z_{2,t}^*\quad.$$

$$-\frac{2}{(\gamma_\upsilon)^2}\left\{\begin{matrix}\left[-\upsilon^2\Gamma_{11}-(\gamma_\upsilon)^2\Gamma_{12}-(1-\upsilon)^2\Gamma_{13}-\frac{\Gamma_{14}}{2}-\upsilon(1-\upsilon)\Gamma_{15}\right]\\+\frac{\Theta_2\gamma_\upsilon\left\{2+\gamma_\upsilon\left[(1-\alpha)+\varphi\right]\right\}}{2}\end{matrix}\right\}n_t a_t$$

Let define:

$$\Gamma_{52}\equiv\left[\upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\Gamma_{14}\right]+\Theta_2\gamma_\upsilon$$

$$\Gamma_{53}\equiv(\gamma_\upsilon)^2\Gamma_{12}+\upsilon(1-\upsilon)\Gamma_{15}$$

$$\Gamma_{54}\equiv-\upsilon\eta(2-\upsilon)(1-\upsilon)2\Gamma_{13}-\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon\right]\Gamma_{15}$$

$$\Gamma_{55}\equiv\eta\upsilon(2-\upsilon)-\gamma_\upsilon$$

$$\Gamma_{56}\equiv2\upsilon\eta(2-\upsilon)-\gamma_\upsilon$$

$$\Gamma_{57}\equiv-\upsilon\eta(2-\upsilon)(1-\upsilon)2\Gamma_{13}-\upsilon\Gamma_{56}\Gamma_{15}$$

$$\Gamma_{57B} \equiv -2(1-\beta)\left(\upsilon^2\Gamma_{11}+\Gamma_{14}\right)$$

$$\Gamma_{58} \equiv \frac{\Gamma_{57}(1-\upsilon)+\Gamma_{55}\Gamma_{57B}}{2(1-\upsilon)}$$

$$\Gamma_{59} \equiv \frac{\Gamma_{55}\upsilon\gamma_\upsilon\Theta_2+\upsilon\gamma_\upsilon\Theta_2\eta(1-\upsilon)(2-\upsilon)}{2(1-\upsilon)}$$

$$\Gamma_{60} \equiv \upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}+\upsilon(1-\upsilon)\Gamma_{15}$$

$$\Gamma_{61} \equiv \frac{\gamma_\upsilon[\eta\Theta_3-\Theta_2]}{2}$$

$$\Gamma_{62} \equiv \frac{\Gamma_{55}\left(\upsilon^2\Gamma_{11}+\Gamma_{14}\right)+(1-\upsilon)^2\upsilon\eta(2-\upsilon)\Gamma_{13}}{(1-\upsilon)}$$

$$\Gamma_{64} \equiv \frac{(1-\upsilon)(2\Gamma_{62}+\upsilon\Gamma_{56}\Gamma_{15})}{2(1-\upsilon)}$$

$$\Gamma_{65} \equiv \frac{\gamma_\upsilon\Theta_2[\eta(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)2}$$

$$\Gamma_{69} \equiv -\upsilon^2\Gamma_{11}-(\gamma_\upsilon)^2\Gamma_{12}-(1-\upsilon)^2\Gamma_{13}-\frac{\Gamma_{14}}{2}-\upsilon(1-\upsilon)\Gamma_{15}$$

$$\Gamma_{70} \equiv \frac{\Theta_2\gamma_\upsilon[2+\gamma_\upsilon(1+\varphi)]}{2}$$

Then, the previous expression can be rewritten as:

$$\begin{aligned}&\frac{1}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{52}+\Gamma_{53}]n_t^2-\frac{2}{\gamma_\upsilon^2\sigma_C}[\Gamma_{58}-\Gamma_{59}]n_tz_t-\frac{2\upsilon}{\gamma_\upsilon^2\sigma_C}[\Gamma_{60}-\Gamma_{61}]n_tz_{1,t}^*\\&-\frac{2}{\gamma_\upsilon^2\sigma_C}[\Gamma_{64}+\Gamma_{65}]n_tz_{2,t}^*-\frac{2}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{69}+\Gamma_{70}]n_ta_t\end{aligned}. \quad (4\text{-}13\text{-}35c)$$

Plugging Eq. (4-13-35c) into Eq.(4-13-35) yields:

$$\sum_{t=0}^{\infty}\beta^t E_0\left\{\Phi y_t-\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}$$

$$=\sum_{t=0}^{\infty}\beta^t E_0\left(\begin{array}{l}\frac{1}{\gamma_\upsilon^2}[\Gamma_{52}+\Gamma_{53}]n_t^2-\frac{2}{\gamma_\upsilon^2}[\Gamma_{58}-\Gamma_{59}]n_t z_t-\frac{2\upsilon}{\gamma_\upsilon^2}[\Gamma_{60}-\Gamma_{61}]n_t z_{1,t}^*\\-\frac{2}{\gamma_\upsilon^2}[\Gamma_{64}+\Gamma_{65}]n_t z_{2,t}^*-\frac{2}{\gamma_\upsilon^2}[\Gamma_{69}+\Gamma_{70}]n_t a_t\\+\frac{\Theta_2\varphi(2+\varphi)}{2}n_t^2\\+\frac{\Theta_2\varepsilon_p[\kappa_p+\varphi]}{(1-\alpha)2\kappa_p}\pi_{H,t}^2+\frac{\Theta_2\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{array}\right),$$

$$+\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}$$

$$+o\left(\|\xi\|^2\right)$$

which can be rewritten as:

$$\sum_{t=0}^{\infty}\beta^t E_0\left\{\Phi y_t-\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}$$

$$=\sum_{t=0}^{\infty}\beta^t E_0\left(\begin{array}{l}\frac{1}{\gamma_\upsilon^2}[\Gamma_{52}+\Gamma_{53}+\Gamma_{72}]n_t^2-\frac{2}{\gamma_\upsilon^2}[\Gamma_{58}-\Gamma_{59}]n_t z_t-\frac{2\upsilon}{\gamma_\upsilon^2}[\Gamma_{60}-\Gamma_{61}]n_t z_{1,t}^*\\-\frac{2}{\gamma_\upsilon^2}[\Gamma_{64}+\Gamma_{65}]n_t z_{2,t}^*-\frac{2}{\gamma_\upsilon^2}[\Gamma_{69}+\Gamma_{70}]n_t a_t\\+\frac{\Theta_2\varepsilon_p[\kappa_p+\varphi]}{2\kappa_p}\pi_{H,t}^2+\frac{\Theta_2\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{array}\right), \quad (4\text{-}13\text{-}82)$$

$$+\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}$$

$$+o\left(\|\xi\|^2\right)$$

with $\Gamma_{72}\equiv\frac{\Theta_2\varphi(2+\varphi)(\gamma_\upsilon)^2}{2}$.

Plugging into Eq.(4-13-82) into Eq.(4-2-17) yields:

$$\tilde{\mathcal{W}} = -\sum_{k=0}^{\infty}\beta^t \mathrm{E}_t \left\{ \begin{array}{l} \left(\begin{array}{l} \frac{1}{\gamma_{\upsilon}^2}[-\Gamma_{52}-\Gamma_{53}-\Gamma_{72}]n_t^2 \\ -\left[\begin{array}{l} \frac{2}{\gamma_{\upsilon}^2}[-\Gamma_{58}+\Gamma_{59}]n_t z_t + \frac{2\upsilon}{\gamma_{\upsilon}^2}[-\Gamma_{60}+\Gamma_{61}]n_t z_{1,t}^* \\ +\frac{2}{\gamma_{\upsilon}^2}[-\Gamma_{64}-\Gamma_{65}]n_t z_{2,t}^* + \frac{2}{\gamma_{\upsilon}^2}[-\Gamma_{69}-\Gamma_{70}]n_t a_t \end{array} \right] \end{array} \right) \\ -\frac{\Theta_2 \varepsilon_p [\kappa_p+\varphi]}{2\kappa_p}\pi_{H,t}^2 - \frac{\Theta_2 \varepsilon_w (1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2 \\ +\frac{(1-\Phi)(1+\varphi)}{2}n_t^2 \\ +\frac{\varepsilon_p(1-\Phi)}{2\kappa_p}\pi_{H,t}^2 + \frac{\varepsilon_w(1+\varepsilon_w\varphi)(1-\Phi)}{2\kappa_w}\left(\pi_t^W\right)^2 \end{array} \right\} + o\left(\|\xi\|^3\right),$$

$$\tilde{\mathcal{W}} = \sum_{k=0}^{\infty}\beta^t \mathrm{E}_t \left\{ \begin{array}{l} \Phi y_{t+k} - \frac{\eta\upsilon(2-\upsilon)}{1-\upsilon}s_{t+k} - \frac{(1-\Phi)(1+\varphi)}{2}n_{H,t+k}^2 \\ -\frac{\varepsilon_p(1-\Phi)}{2\kappa_p}\pi_{H,t+k}^2 - \frac{\varepsilon_w(1+\varepsilon_w\varphi)(1-\Phi)}{2\kappa_w}\left(\pi_{t+k}^W\right)^2 \end{array} \right\} + o\left(\|\xi\|^3\right), \quad (4\text{-}2\text{-}17)$$

which can be rewritten as:

$$\tilde{\mathcal{W}} = -\sum_{k=0}^{\infty}\beta^t \mathrm{E}_t \left\{ \begin{array}{l} \left(\begin{array}{l} \frac{1}{\gamma_{\upsilon}^2}\left[-\Gamma_{52}-\Gamma_{53}-\Gamma_{72}+\frac{\gamma_{\upsilon}^2(1-\Phi)(1+\varphi)}{2}\right]n_t^2 \\ -\left[\begin{array}{l} \frac{2}{\gamma_{\upsilon}^2}[-\Gamma_{58}+\Gamma_{59}]n_t z_t + \frac{2\upsilon}{\gamma_{\upsilon}^2}[-\Gamma_{60}+\Gamma_{61}]n_t z_{1,t}^* \\ +\frac{2}{\gamma_{\upsilon}^2}[-\Gamma_{64}-\Gamma_{65}]n_t z_{2,t}^* + \frac{2}{\gamma_{\upsilon}^2}[-\Gamma_{69}-\Gamma_{70}]n_t a_t \end{array} \right] \end{array} \right) \\ +\frac{\varepsilon_p\left\{(1-\Phi)-\Theta_2[\kappa_p+\varphi]\right\}}{2\kappa_p}\pi_{H,t}^2 \\ +\frac{\varepsilon_w\left[-\Theta_2(1+\varphi)\kappa_w+(1+\varepsilon_w\varphi)(1-\Phi)\right]}{2\kappa_w}\left(\pi_t^W\right)^2 \end{array} \right\} + o\left(\|\xi\|^3\right)$$

,

In the previous expression, line 1 can be rewritten as:

$$\frac{1}{\gamma_\upsilon^2}\begin{bmatrix}-\Gamma_{52}-\Gamma_{53}-\Gamma_{72}\\+\frac{\gamma_\upsilon^2(1-\Phi)(1+\varphi)}{2}\end{bmatrix}n_t^2-\begin{bmatrix}\frac{2}{\gamma_\upsilon^2}[-\Gamma_{58}+\Gamma_{59}]n_t z_t+\frac{2\upsilon}{\gamma_\upsilon^2}[-\Gamma_{60}+\Gamma_{61}]n_t z_{1,t}^*\\+\frac{2}{\gamma_\upsilon^2}[-\Gamma_{64}-\Gamma_{65}]n_t z_{2,t}^*+\frac{2}{\gamma_\upsilon^2}[-\Gamma_{69}-\Gamma_{70}]n_t a_t\end{bmatrix}.$$

$$=\frac{1}{\gamma_\upsilon^2}\begin{bmatrix}-\Gamma_{52}-\Gamma_{53}-\Gamma_{72}\\+\frac{\gamma_\upsilon^2(1-\Phi)(1+\varphi)}{2}\end{bmatrix}n_t^2-2n_t\begin{bmatrix}\frac{-\Gamma_{58}+\Gamma_{59}}{\gamma_\upsilon^2}z_t+\frac{\upsilon[-\Gamma_{60}+\Gamma_{61}]}{\gamma_\upsilon^2}z_{1,t}^*\\+\frac{-\Gamma_{64}-\Gamma_{65}}{\gamma_\upsilon^2}z_{2,t}^*+\frac{-\Gamma_{69}-\Gamma_{70}}{\gamma_\upsilon^2}a_t\end{bmatrix}$$

Let define $\Omega_0\equiv-\Gamma_{52}-\Gamma_{53}-\Gamma_{72}+\frac{\gamma_\upsilon^2(1-\Phi)(1+\varphi)}{2}$. Then:

$$\frac{\Omega_0}{\gamma_\upsilon^2}n_t^2-2n_t\left[\frac{-\Gamma_{58}+\Gamma_{59}}{\gamma_\upsilon^2}z_t+\frac{\upsilon[-\Gamma_{60}+\Gamma_{61}]}{\gamma_\upsilon^2}z_{1,t}^*+\frac{-\Gamma_{64}-\Gamma_{65}}{\gamma_\upsilon^2}z_{2,t}^*+\frac{-\Gamma_{69}-\Gamma_{70}}{\gamma_\upsilon^2}a_t\right]$$

$$=\frac{\Omega_0}{\gamma_\upsilon^2}n_t^2\left\{n_t^2-2\frac{\gamma_\upsilon^2}{\Omega_0}n_t\begin{bmatrix}\frac{-\Gamma_{58}+\Gamma_{59}}{\gamma_\upsilon^2}z_t+\frac{\upsilon[-\Gamma_{60}+\Gamma_{61}]}{\gamma_\upsilon^2}z_{1,t}^*+\frac{-\Gamma_{64}-\Gamma_{65}}{\gamma_\upsilon^2}z_{2,t}^*\\+\frac{-\Gamma_{69}-\Gamma_{70}}{\gamma_\upsilon^2}a_t\end{bmatrix}\right\},$$

$$=\frac{\Omega_0}{\gamma_\upsilon^2}\left\{n_t^2-2n_t\left[\frac{-\Gamma_{58}+\Gamma_{59}}{\Omega_0}z_t+\frac{\upsilon(-\Gamma_{60}+\Gamma_{61})}{\Omega_0}z_{1,t}^*+\frac{-\Gamma_{64}-\Gamma_{65}}{\Omega_0}z_{2,t}^*+\frac{-\Gamma_{69}-\Gamma_{70}}{\Omega_0}a_t\right]\right\}$$

Let define:

$$\Omega_1\equiv\frac{-\Gamma_{58}+\Gamma_{59}}{\Omega_0}$$

$$\Omega_2\equiv\frac{-\Gamma_{60}+\Gamma_{61}}{\Omega_0}$$

$$\Omega_3\equiv\frac{-\Gamma_{64}-\Gamma_{65}}{\Omega_0}$$

$$\Omega_5\equiv\frac{-\Gamma_{69}-\Gamma_{70}}{\Omega_0}$$

and let define:

$$n_t^e=\Omega_1 z_t+\upsilon\Omega_2 z_{1,t}^*+\Omega_3 z_{2,t}^*+\Omega_5 a_t.$$

Then the previous expression can be rewritten as:

Finally, we have:

$$\tilde{\mathcal{W}} = -\sum_{k=0}^{\infty}\beta^t \mathrm{E}_t \left\{ \begin{aligned} &\frac{\Omega_0}{\gamma_\upsilon^2}\left(n_t - n_t^e\right)^2 \\ &+\frac{\varepsilon_p\left[(1-\Phi)-\Theta_2\left(\kappa_p+\varphi\right)\right]}{2\kappa_p}\pi_{H,t}^2 \\ &+\frac{\varepsilon_w\left[-\Theta_2(1+\varphi)\kappa_w+(1+\varepsilon_w\varphi)(1-\Phi)\right]}{2\kappa_w}\left(\pi_t^W\right)^2 \end{aligned} \right\} + \text{s.o.t.i.p.} + o\left(\|\xi\|^3\right).$$

Let define $\Lambda_n \equiv \frac{2\Omega_0}{\gamma_\upsilon^2}$, $\Lambda_p \equiv \frac{\varepsilon_p\left[(1-\Phi)-\Theta_2\left(\kappa_p+\varphi\right)\right]}{\kappa_p}$ and $\Lambda_w \equiv \frac{\varepsilon_w\left[-\Theta_2(1+\varphi)\kappa_w+(1+\varepsilon_w\varphi)(1-\Phi)\right]}{\kappa_w}$. Then we have:

$$\tilde{\mathcal{W}} = -\sum_{k=0}^{\infty}\beta^t \mathrm{E}_t\left[\frac{\Lambda_n}{2}\left(n_t-n_t^e\right)^2+\frac{\Lambda_p}{2}\pi_{H,t}^2+\frac{\Lambda_w}{2}\left(\pi_t^W\right)^2\right]+\text{s.o.t.i.p.}+o\left(\|\xi\|^3\right),$$

Let define $\hat{n}_t \equiv n_t - n_t^e$. Then we get.

$$\tilde{\mathcal{W}} = -\sum_{k=0}^{\infty}\beta^t \mathrm{E}_t\left[\frac{\Lambda_n}{2}\hat{n}_t^2+\frac{\Lambda_p}{2}\pi_{H,t}^2+\frac{\Lambda_w}{2}\left(\pi_t^W\right)^2\right]+\text{s.o.t.i.p.}+o\left(\|\xi\|^3\right).$$

Based on the previous expression, we have:

$$L \sim \frac{\Lambda_n}{2}\mathrm{var}\left(\hat{n}_t\right)+\frac{\Lambda_p}{2}\mathrm{var}\left(\pi_{H,t}\right)+\frac{\Lambda_w}{2}\mathrm{var}\left(\pi_t^W\right)^2,$$

which is Eq.(15) in the text.

5 The NKPC and the Wage PC with Efficient Level

Eq.(3-1-34) can be rewritten as:

$$\begin{aligned} \mu_t^w &= w_t - p_{H,t} - \upsilon s_t - \varphi n_t - c_t \\ &= w_t - p_t + p_t - p_{H,t} - \upsilon s_t - \varphi n_t - c_t \\ &= \omega_t - \left(p_{H,t} - p_t\right) - \upsilon s_t - \varphi n_t - c_t \\ &= \omega_t - \varphi n_t - c_t \end{aligned}, \quad (6\text{-}1)$$

with $\omega_t \equiv d\frac{W_t}{P_t}\Big/\frac{W}{P}$ being the real (consumption) wage. Under the flexible wage equilibrium, $\mu_t^W = 0$ is applied. Under that equilibrium, Eq.(6-9) can be rewritten as:

$\omega_t^e = \varphi n_t^e + c_t^e$, (6-1)

with $\omega_t^e \equiv \omega_t - \hat{\omega}_t$ and $c_t^e \equiv c_t - \hat{c}_t$.

Eq.(3-1-24) can be rewritten as:

$$\begin{aligned} mc_t &= \frac{1}{1-\tau}\tau_t + w_t - p_{H,t} - p_t + p_t - a_t - \frac{1}{1-\tau}\tau \\ &= \frac{1}{1-\tau}\tau_t + \omega_t - \left(p_{H,t} - p_t\right) - a_t - \frac{1}{1-\tau}\tau \\ &= \frac{1}{1-\tau}\tau_t + \omega_t - x_{H,t} - a_t - \frac{1}{1-\tau}\tau \\ &= \frac{1}{1-\tau}\tau_t + \omega_t + \upsilon s_t - a_t - \frac{1}{1-\tau}\tau \end{aligned} \text{, (6-2)}$$

where we use Eq.(3-1-7).

with $s_t^e \equiv s_t - \hat{s}_t$.

Eq.(3-1-11) can be rewritten as:

$$s_t = \frac{1}{1-\upsilon}c_t - \frac{1}{1-\upsilon}z_t + \frac{1}{1-\upsilon}z_{2,t}^*.$$

which can be rewritten as:

$$s_t^e = \frac{1}{1-\upsilon}c_t^e - \frac{1}{1-\upsilon}z_t + \frac{1}{1-\upsilon}z_{2,t}^*. \text{ (6-4)}$$

under the efficient equilibrium.

Eq.(3-1-8) can be rewritten as:

$$\begin{aligned} c_t &= \frac{1}{1-\upsilon}y_t - \frac{\eta\upsilon(2-\upsilon)}{1-\upsilon}s_t - \frac{\upsilon}{1-\upsilon}z_{1,t}^* \\ &= \frac{1}{1-\upsilon}y_t - \frac{\eta\upsilon(2-\upsilon)}{1-\upsilon}\left(\frac{1}{1-\upsilon}c_t - \frac{1}{1-\upsilon}z_t + \frac{1}{1-\upsilon}z_{2,t}^*\right) - \frac{\upsilon}{1-\upsilon}z_{1,t}^* \\ &= \frac{1}{1-\upsilon}y_t - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}c_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_t - \frac{\upsilon}{1-\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_{2,t}^* \end{aligned},$$

where we use Eq.(3-1-11). The previous expression can be rewritten as:

$$\frac{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}c_t = \frac{1}{1-\upsilon}y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_t - \frac{\upsilon}{1-\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_{2,t}^*.$$

Then, we have:

$$c_t = \frac{(1-\upsilon)^2}{\gamma_\upsilon}\left[\frac{1}{1-\upsilon}y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_t - \frac{\upsilon}{1-\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_{2,t}^*\right] \\ = \frac{1-\upsilon}{\gamma_\upsilon}y_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_{2,t}^*,$$

with $\gamma_\upsilon \equiv (1-\upsilon)^2 + \eta\upsilon(2-\upsilon)$.

The previous expression can be rewritten as:

$$c_t^e = \frac{1-\upsilon}{\gamma_\upsilon}y_t^e + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_{2,t}^*, \quad (6\text{-}5)$$

under the efficient equilibrium.

Eq.(3-1-26) can be rewritten as:

$$y_t^e = n_t^e + a_t, \quad (6\text{-}6)$$

under the efficient equilibrium.

Plugging Eq.(6-6) into Eq.(6-5) yields:

$$c_t^e = \frac{1-\upsilon}{\gamma_\upsilon}n_t^e + \frac{1-\upsilon}{\gamma_\upsilon}a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_{2,t}^*. \quad (6\text{-}7)$$

Plugging Eq.(6-7) into Eq.(6-1) yields:

$$\omega_t^e = \varphi n_t^e + \frac{1-\upsilon}{\gamma_\upsilon}n_t^e + \frac{1-\upsilon}{\gamma_\upsilon}a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_{2,t}^* \\ = \frac{\varphi\gamma_\upsilon + 1-\upsilon}{\gamma_\upsilon}n_t^e + \frac{1-\upsilon}{\gamma_\upsilon}a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_{2,t}^*. \quad (6\text{-}8)$$

Plugging Eq.(6-7) into Eq.(6-4) yields:

$$s_t^e = \frac{1}{1-\upsilon}\left[\frac{1-\upsilon}{\gamma_\upsilon}n_t^e + \frac{1-\upsilon}{\gamma_\upsilon}a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_{2,t}^*\right] \\ - \frac{1}{1-\upsilon}z_t + \frac{1}{1-\upsilon}z_{2,t}^* \\ = \frac{1}{\gamma_\upsilon}n_t^e + \frac{1}{\gamma_\upsilon}a_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon}z_t - \frac{\upsilon}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon}z_{2,t}^*. \quad (6\text{-}9)$$

Now, we turn to Eq.(6-1) $\mu_t^w = \omega_t - \varphi n_t - c_t$. Plugging Eq.(3-1-26) into Eq.(6-1) yields:

$$\mu_t^w = \omega_t - \varphi y_t - c_t + \varphi a_t \\ = \hat{\omega}_t - \varphi\hat{y}_t - \hat{c}_t + \omega_t^e - \varphi y_t^e - c_t^e + \varphi a_t.$$

Plugging Eqs.(6-6), (6-7) and (6-8) into the previous expression yields:

$$
\begin{aligned}
\mu_t^w &= \hat{\omega}_t - \varphi \hat{y}_t - \hat{c}_t + \left[\frac{\varphi\gamma_\upsilon + 1 - \upsilon}{\gamma_\upsilon} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \right] \\
&\quad -\varphi\left(n_t^e + a_t\right) \\
&\quad -\left[\frac{1-\upsilon}{\gamma_\upsilon} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \right] + \varphi a_t \\
&= \hat{\omega}_t - \varphi \hat{y}_t - \hat{c}_t + \frac{\varphi\gamma_\upsilon + 1 - \upsilon - \gamma_\upsilon\varphi - (1-\upsilon)}{\gamma_\upsilon} n_t^e \\
&= \hat{\omega}_t - \varphi \hat{y}_t - \hat{c}_t
\end{aligned}
$$

Then, we have:

$\mu_t^w = \hat{\omega}_t - \varphi \hat{y}_t - \hat{c}_t$. (6-14) [(27) in the text]

Eqs. (6-8) and (6-9) into Eq.(6-2) yields:

$$
\begin{aligned}
mc_t &= \hat{\omega}_t + \upsilon \hat{s}_t + \omega_t^e + \upsilon s_t^e - a_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&= \hat{\omega}_t + \upsilon \hat{s}_t \\
&\quad + \frac{\varphi\gamma_\upsilon + 1 - \upsilon}{\gamma_\upsilon} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \\
&\quad + \upsilon\left[\frac{1}{\gamma_\upsilon} n_t^e + \frac{1}{\gamma_\upsilon} a_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^*\right] \\
&\quad - a_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&= \hat{\omega}_t + \upsilon \hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&\quad + \frac{\varphi\gamma_\upsilon + 1 - \upsilon}{\gamma_\upsilon} n_t^e + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t + \frac{\upsilon\left[\eta(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\upsilon^2}{\gamma_\upsilon} z_{1,t}^* \\
&\quad - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* - a_t \\
&= \hat{\omega}_t + \upsilon \hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t \\
&\quad + \frac{1+\varphi\gamma_\upsilon}{\gamma_\upsilon} n_t^e + \frac{\upsilon\left\{\eta(2-\upsilon)(1-\upsilon)+\eta(2-\upsilon)-\gamma_\upsilon\right\}}{\gamma_\upsilon(1-\upsilon)} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* \\
&\quad - \frac{\upsilon\left\{\eta(2-\upsilon)+\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right\}}{\gamma_\upsilon(1-\upsilon)} z_{2,t}^* \\
&= \hat{\omega}_t + \upsilon \hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t + \frac{1+\varphi\gamma_\upsilon}{\gamma_\upsilon} n_t^e \\
&\quad + \frac{\upsilon\left\{\eta(2-\upsilon)\left[(1-\upsilon)+1\right]-\gamma_\upsilon\right\}}{\gamma_\upsilon(1-\upsilon)} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* \\
&\quad - \frac{\upsilon\left\{\eta(2-\upsilon)\left[(1-\upsilon)+1\right]-\gamma_\upsilon\right\}}{\gamma_\upsilon(1-\upsilon)} z_{2,t}^* \\
&= \hat{\omega}_t + \upsilon \hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t + \frac{1+\varphi\gamma_\upsilon}{\gamma_\upsilon} n_t^e \\
&\quad + \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* - \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)} z_{2,t}^* \quad .
\end{aligned}
$$

Then, we have:

$$mc_t = \hat{\omega}_t + \upsilon\hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t + \frac{1+\varphi\gamma_\upsilon}{\gamma_\upsilon}n_t^e + \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_t - \frac{\upsilon}{\gamma_\upsilon}z_{1,t}^* - \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_{2,t}^* . \quad (6\text{-}15)$$

Adding the LHS of Eq.(3-1-12) $y_t = \left[(\eta-1)\upsilon(2-\upsilon)+1\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^*$ while subtracting the RHS of Eq.(3-1-12) into Eq.(6-15) yields:

$$\begin{aligned}
mc_t &= \hat{\omega}_t + \upsilon\hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t + \frac{1+\varphi\gamma_\upsilon}{\gamma_\upsilon}n_t^e + \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_t \\
&\quad - \frac{\upsilon}{\gamma_\upsilon}z_{1,t}^* - \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_{2,t}^* \\
&= \hat{\omega}_t + \upsilon\hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t + \frac{1+\varphi\gamma_\upsilon}{\gamma_\upsilon}n_t^e + \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_t \\
&\quad - \frac{\upsilon}{\gamma_\upsilon}z_{1,t}^* - \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_{2,t}^* \\
&\quad + \left\{\hat{y}_t + y_t^e - \left[(\eta-1)\upsilon(2-\upsilon)+1\right](\hat{s}_t + s_t^e) - (1-\upsilon)z_t - \upsilon z_{1,t}^* + (1-\upsilon)z_{2,t}^*\right\} \\
&= \hat{\omega}_t + \hat{y}_t + \left\{\upsilon\left[1-(\eta-1)(2-\upsilon)\right]+1\right\}\hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau + \frac{1+\varphi\gamma_\upsilon+\gamma_\upsilon}{\gamma_\upsilon}n_t^e \\
&\quad - \left[(\eta-1)\upsilon(2-\upsilon)+1\right]s_t^e + \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]-\gamma_\upsilon(1-\upsilon)^2}{\gamma_\upsilon(1-\upsilon)}z_t - \frac{\upsilon(1+\gamma_\upsilon)}{\gamma_\upsilon}z_{1,t}^* \\
&\quad - \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]-\gamma_\upsilon(1-\upsilon)^2}{\gamma_\upsilon(1-\upsilon)}z_{2,t}^* \\
&= \hat{\omega}_t + \hat{y}_t + \left\{\upsilon\left[1-(\eta-1)(2-\upsilon)\right]+1\right\}\hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau + \frac{1+\gamma_\upsilon(1+\varphi)}{\gamma_\upsilon}n_t^e \\
&\quad - \left[(\eta-1)\upsilon(2-\upsilon)+1\right]s_t^e + \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]-\gamma_\upsilon(1-\upsilon)^2}{\gamma_\upsilon(1-\upsilon)}z_t - \frac{\upsilon(1+\gamma_\upsilon)}{\gamma_\upsilon}z_{1,t}^* \\
&\quad - \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]-\gamma_\upsilon(1-\upsilon)^2}{\gamma_\upsilon(1-\upsilon)}z_{2,t}^* , \quad (6\text{-}16)\ [(22)\text{ in the text}]
\end{aligned}$$

where we use Eq.(6-6) $y_t^e = n_t^e + a_t$ to eliminate y_t^e in the 5th line.

Chapter 2 The FTPL Model

1 The Model

1.2 FTPL Model

1.2.8 FTPL and the GBC with Euler Equation

The GBC is given by:

$B_t^n = R_{t-1}B_{t-1}^n - P_t SP_t$, (1-1-45) [(37) in the text]

with

$SP_t \equiv \frac{P_{H,t}}{P_t}\left(\tau_t Y_t - G_t\right) - \zeta_{H,t}$. (1-1-46) [(38) in the text (Please ignore $\zeta_{H,t}$)]

Multiplying R_t both sides of Eq.(1-1-45) yields:

$R_t B_t^n = R_t R_{t-1} B_{t-1}^n - R_t P_t SP_t$

Iterating *j* times implies:

$$R_{t+j}B_{t+j}^n = \prod_{h=0}^{j} R_{t+h}B_{t-1}^n - \sum_{h=0}^{j}\left(\prod_{k=h}^{j} R_{t+k}\right)P_{t+h}SP_{t+h} \quad \text{(1-1-54)}$$

Dividing both sides of Eq.(1-1-54) by P_{t+j+1} yields:

$$R_{t+j}\frac{B_{t+j}^n}{P_{t+j+1}} = \frac{1}{P_{t+j+1}}\prod_{h=0}^{j} R_{t+h}B_{t-1}^n - \frac{1}{P_{t+j+1}}\sum_{h=0}^{j}\left(\prod_{k=h}^{j} R_{t+k}\right)P_{t+h}SP_{H,t+h} \quad \text{(1-1-55)}$$

The first term on the RHS in Eq.(1-1-55) can be rewritten as:

$$\begin{aligned}
\frac{1}{P_{t+j+1}}\prod_{h=0}^{j} R_{t+h}B_{H,t-1}^n &= \frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}\frac{P_{t+j-2}}{P_{t+j-1}}\cdots\frac{P_t}{P_{t+1}}R_{t+j}R_{t+j-1}R_{t+j-2}\cdots\frac{B_{H,t-1}^n}{P_t} \\
&= R_{t+j}\frac{P_{t+j}}{P_{t+j+1}}R_{t+j-1}\frac{P_{t+j-1}}{P_{t+j}}\cdots R_t\frac{P_t}{P_{t+1}}\frac{B_{t-1}^n}{P_t} \qquad \text{(1-56)} \\
&= \left(\prod_{h=0}^{j} R_{t+h}\frac{P_{t+j}}{P_{t+j+1}}\right)\frac{B_{t-1}^n}{P_t}
\end{aligned}$$

The second term on the RHS in Eq.(1-55) can be rewritten as:

$$
\frac{1}{P_{t+j+1}}\sum_{h=0}^{j}\left(\prod_{k=h}^{j}R_{t+k}\right)P_{t+h}SP_{H,t+h}=\frac{1}{P_{t+j+1}}\begin{pmatrix}\prod_{k=0}^{j}R_{t+k}P_tSP_{H,t}\\+\prod_{k=1}^{j}R_{t+k}P_{t+1}SP_{H,t+1}+\cdots\\+\prod_{k=j-1}^{j}R_{t+k}P_{t+j-1}SP_{H,t+j-1}\\+R_{t+j}P_{t+j}SP_{H,t+j}\end{pmatrix}
$$

$$
\begin{aligned}
&=R_{t+j}\frac{P_{t+j}}{P_{t+j+1}}SP_{H,t+j}+\prod_{k=j-1}^{j}R_{t+k}\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}SP_{H,t+j-1}\\
&+\prod_{k=j-2}^{j}R_{t+k}\frac{P_{t+j}}{P_{t+j+1}}\frac{P_{t+j-1}}{P_{t+j}}\frac{P_{t+j-2}}{P_{t+j-1}}SP_{H,t+j-2}+\cdots+\prod_{k=0}^{j}R_{t+k}\frac{P_{t+j}}{P_{t+j+1}}\cdots\frac{P_t}{P_{t+1}}SP_{H,t}\\
&=\prod_{k=j}^{j}R_{t+k}\Pi_{t+k+1}^{-1}SP_{H,t+j}+\prod_{k=j-1}^{j}R_{t+k}\Pi_{t+k+1}^{-1}SP_{H,t+j-1}\cdots\\
&+\prod_{k=1}^{j}R_{t+k}\Pi_{t+k+1}^{-1}SP_{H,t+1}+\prod_{k=0}^{j}R_{t+k}\Pi_{t+k+1}^{-1}SP_{H,t}\\
&=\sum_{h=0}^{j}\left(\prod_{k=h}^{j}R_{t+k}\Pi_{t+k+1}^{-1}\right)SP_{H,t+h}
\end{aligned}
\quad . \text{(1-1-57)}
$$

Plugging Eqs.(1-1-56) and (1-1-57) into Eq.(1-1-55) yields:

$$
R_{t+j}\frac{B_{H,t+j}^{n}}{P_{t+j+1}}=\left(\prod_{h=0}^{j}R_{t+h}\Pi_{t+j+1}^{-1}\right)\frac{B_{t-1}^{n}}{P_t}+\sum_{h=0}^{j}\left(\prod_{k=h}^{j}R_{t+k}\Pi_{t+k+1}^{-1}\right)SP_{t+h}\ . \text{(1-1-58)}
$$

Eq.(1-1-7) can be rewritten as:

$$
\Pi_{t+1}^{-1}=\frac{\beta^{-1}}{R_t}\frac{C_t^{-1}}{C_{t+1}^{-1}}\frac{Z_t}{Z_{t+1}} \quad \text{(1-1-59).}
$$

Plugging Eq.(1-1-59) into Eq.(1-1-56) yields:

$$
\begin{aligned}
\frac{1}{P_{t+j+1}}\prod_{h=0}^{j}R_{t+h}B_{t-1}^{n}&=R_{t+j}\frac{\beta^{-1}}{R_{t+j}}\frac{C_{t+j}^{-1}}{C_{t+j+1}^{-1}}\frac{Z_{t+j}}{Z_{t+j+1}}R_{t+j-1}\frac{\beta^{-1}}{R_{t+j-1}}\frac{C_{t+j-1}^{-1}}{C_{t+j}^{-1}}\frac{Z_{t+j-1}}{Z_{t+j}}\\
&\cdots R_{t+1}\frac{\beta^{-1}}{R_{t+1}}\frac{C_{t+1}^{-1}}{C_{t+2}^{-1}}\frac{Z_{t+1}}{Z_{t+2}}R_t\frac{\beta^{-1}}{R_t}\frac{C_t^{-1}}{C_{t+1}^{-1}}\frac{B_{t-1}^{n}}{P_t}\frac{Z_t}{Z_{t+1}} \qquad \text{(1-1-60)}\\
&=\beta^{-(j+1)}\frac{C_t^{-1}}{C_{t+j+1}^{-1}}\frac{B_{t-1}^{n}}{P_t}\frac{Z_t}{Z_{t+j+1}}
\end{aligned}
$$

Plugging Eq.(1-1-59) into Eq.(1-1-57) yields:

$$\frac{1}{P_{t+j+1}}\sum_{h=0}^{j}\left(\prod_{k=h}^{j}R_{t+k}\right)P_{t+h}SP_{t+h}$$

$$=\sum_{h=0}^{j}\left(\prod_{k=h}^{j}R_{t+k}\frac{\beta^{-1}}{R_{t+j}}\frac{C_t^{-1}}{C_{t+1}^{-1}}\frac{Z_t}{Z_{t+1}}\right)SP_{t+h}$$

$$=\beta^{-1}\frac{C_{t+j}^{-1}}{C_{t+j+1}^{-1}}\frac{Z_{t+j}}{Z_{t+j+1}}SP_{t+j}+\beta^{-2}\frac{C_{t+j-1}^{-1}}{C_{t+j+1}^{-1}}\frac{Z_{t+j-1}}{Z_{t+j}}SP_{t+j-1}+\cdots+\beta^{-j}\frac{C_{t+1}^{-1}}{C_{t+j+1}^{-1}}\frac{Z_{t+j}}{Z_{t+j+1}}SP_{t+1}+\beta^{-j-1}\frac{C_t^{-1}}{C_{t+1}^{-1}}\frac{Z_t}{Z_{t+1}}SP_t$$

(1-1-61)

Plugging Eqs.(1-1-60) and (1-1-61) into Eq.(1-1-55) yields:

$$R_{t+j}\frac{B_{t+j}^n}{P_{t+j+1}}=\beta^{-(j+1)}\frac{C_t^{-1}}{C_{t+j+1}^{-1}}\frac{Z_t}{Z_{t+j+1}}\frac{B_{t-1}^n}{P_t}-\frac{1}{C_{t+j+1}^{-1}Z_{t+j+1}}\sum_{h=0}^{j}\beta^{h-j-1}C_{t+h}^{-1}Z_{t+h}SP_{t+h} \quad (1\text{-}1\text{-}62)$$

Multiplying β^j on both sides of Eq.(1-1-62) yields:

$$\beta^jR_{t+j}\frac{B_{t+j}^n}{P_{t+j+1}}=\beta^{-1}\frac{C_t^{-1}}{C_{t+j+1}^{-1}}\frac{Z_t}{Z_{t+j+1}}\frac{B_{t-1}^n}{P_t}-\frac{1}{C_{t+j+1}^{-1}Z_{t+j+1}}\sum_{h=0}^{j}\beta^{h-1}C_{t+h}^{-1}Z_{t+h}SP_{t+h} \quad (1\text{-}1\text{-}63)$$

Take the limit for $j\to\infty$ yields:

$$0=\beta^{-1}C_t^{-1}Z_t\frac{B_{t-1}^n}{P_t}-\sum_{h=0}^{\infty}\beta^{h-1}C_{t+h}^{-1}Z_{t+h}SP_{t+h} \quad (1\text{-}1\text{-}64)$$

Here, the TVC is given by:

$$\lim_{j\to\infty}\beta^jR_{t+j}\frac{B_{t+j}^n}{P_{t+j+1}}=0$$

Eq.(1-1-64) can be rewritten as:

$$\beta^{-1}C_t^{-1}Z_t\frac{B_{t-1}^n}{P_t}=\sum_{h=0}^{\infty}\beta^{h-1}C_{t+h}^{-1}Z_{t+h}SP_{t+h}$$
$$=\beta^{-1}C_t^{-1}Z_tSP_t+C_{t+1}^{-1}Z_{t+1}SP_{t+1}+\beta C_{t+2}^{-1}Z_{t+2}SP_{t+2}+\cdots$$

Multiplying β on both sides of the previous expression yields:

$$C_t^{-1}Z_t\frac{B_{t-1}^n}{P_t}=\sum_{h=0}^{\infty}\beta^hC_{t+h}^{-1}Z_{t+h}SP_{t+h} \quad (1\text{-}1\text{-}65)$$
$$=C_t^{-1}Z_tSP_t+\beta C_{t+1}^{-1}Z_{t+1}SP_{t+1}+\beta^2C_{t+2}^{-1}Z_{t+2}SP_{t+2}+\cdots$$

which can be rewritten as:

$$1=\frac{\sum_{k=0}^{\infty}\beta^k\mathrm{E}_t\left(C_{t+k}^{-1}Z_{t+k}SP_{t+k}\right)}{C_t^{-1}Z_t\frac{B_{t-1}^n}{P_t}} \quad \text{or} \quad 1=\frac{\sum_{k=0}^{\infty}\beta^k\mathrm{E}_t\left(C_{t+k}^{-1}Z_{t+k}SP_{t+k}\right)}{C_t^{-1}D_tB_{t-1}\Pi_t^{-1}} \quad \text{[(40) in the text]}$$

Leading Eq.(1-1-65) one period yields:

$$C_{t+1}^{-1}Z_{t+1}\frac{B_t^n}{P_{t+1}}=C_{t+1}^{-1}Z_{t+1}SP_{t+1}+\beta C_{t+2}^{-1}Z_{t+2}SP_{t+2}+\beta^2C_{t+3}^{-1}Z_{t+3}SP_{t+3}+\cdots$$

Multiplying β on both sides of the previous expression yields:

$$\beta C_{t+1}^{-1}Z_{t+1}\frac{B_t^n}{P_{t+1}}=\beta C_{t+1}^{-1}Z_{t+1}SP_{t+1}+\beta^2C_{t+2}^{-1}Z_{t+2}SP_{t+2}+\beta^3C_{t+3}^{-1}Z_{t+3}SP_{t+3}+\cdots \quad (1\text{-}1\text{-}66)$$

Plugging Eq.(1-1-66) into the second line in Eq.(1-1-65) yields:

$$C_t^{-1}Z_t\frac{B_{t-1}^n}{P_t}=C_t^{-1}SP_tZ_t+\beta C_{t+1}^{-1}Z_{t+1}\frac{B_t^n}{P_{t+1}}$$

By dividing $C_t^{-1}Z_t$ yields:

$\frac{B_{t-1}^n}{P_t}=SP_t+\beta\mathrm{E}_t\left(\frac{C_{t+1}^{-1}}{C_t^{-1}}\frac{Z_{t+1}}{Z_t}\frac{B_t^n}{P_{t+1}}\right)$ or $B_{t-1}\Pi_t^{-1}=SP_t+\beta\mathrm{E}_t\left(\frac{C_{t+1}^{-1}}{C_t^{-1}}\frac{Z_{t+1}}{Z_t}B_t\Pi_{t+1}^{-1}\right)$. (1-1-67) [(41) in the text]

1.1.9 Market Clearing Condition

The market clearing conditions in the SOE is given by:

$Y_t(i)=C_t(i)+G_t(i)+EX_t(i)$. (1-1-68)

Let define $Y_t\equiv\left[\int_0^1 Y_t(i)^{\frac{\varepsilon_p-1}{\varepsilon_p}}di\right]^{\frac{\varepsilon_p}{\varepsilon_p-1}}$. Combining them with the definitions of the PPI indices yields:

$Y_t(i)=\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon_p}Y_t$. (1-1-70)

Let define $G_t\equiv\left[\int_0^1 G_t(i)^{\frac{\varepsilon_p-1}{\varepsilon_p}}dh\right]^{\frac{\varepsilon_p}{\varepsilon_p-1}}$. Combining this with the definitions of the PPI indices yields:

$G_t(i)=\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon_p}G_t$, (1-1-72)

Eq.(1-1-43) can be rewritten as:

$$
\begin{aligned}
EX_t &= \upsilon\left(\frac{P^*_{H,t}}{P^*_t}\right)^{-\eta} C^*_t \\
&= \upsilon\left(\frac{P_{H,t}\big/E_t}{P^*_{F,t}}\right)^{-\eta} C^*_t \\
&= \upsilon\left(\frac{P_{H,t}\big/E_t}{P_{F,t}\big/E_t}\right)^{-\eta} C^*_t \text{ , (1-1-43).} \\
&= \upsilon\left(\frac{P_{H,t}}{P_{F,t}}\right)^{-\eta} C^*_t \\
&= \upsilon S^{\eta} C^*_t
\end{aligned}
$$

Plugging Eqs. (1-1-31), (1-1-32), (1-1-39),(1-1-70), (1-1-72) and (1-1-73) into Eq.(1-1-68) yields:

$$
\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon_p} Y_t = \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon_p} \left[\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} (1-\upsilon)C_t + S^{\eta}_t \upsilon C^*_t + G_t\right]
$$

which can be rewritten as:

$$
Y_t = (1-\upsilon)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t + \upsilon S^{\eta}_t C^*_t + G_t \text{ . (1-1-76)}
$$

because $1-\upsilon=\alpha$ and $\upsilon^*=0$.

1.1.10 Firms

Production function is given by:

$$
Y_t(i) = A_t N_t(i) \text{ ,}
$$

with

$$
N_t(i) \equiv \left[\int_0^1 N_t(i,j)^{1-\frac{1}{\varepsilon_w}} dj\right]^{\frac{\varepsilon_w}{\varepsilon_w-1}}
$$

Plugging the production function into Eq.(1-1-70) yields:

$$
N_t(i) = \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} \frac{Y_t}{A_t} \text{ . (1-1-78).}
$$

Integrating the previous expression yields:

$$\int_0^1 N_t(i)di = \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} \frac{Y_t}{A_t} di$$
$$= \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} di \frac{Y_t}{A_t},$$

which can be rewritten as:

$$N_t = \frac{Y_t \Delta_{p,t}}{A_t}, \text{ (1-1-80)}$$

with $N_t = \int_0^1 N_t(i)di$ and:

$$\Delta_{p,t} = \int_0^1 \left(\frac{P_t(h)}{P_{H,t}}\right)^{-\varepsilon} dh. \text{ (1-1-81)}$$

The FONC for firms is given by:

$$\tilde{P}_{H,t} = \frac{\frac{\varepsilon_p}{\varepsilon_p - 1} \sum_{k=0}^{\infty} (\beta\theta)^k \mathrm{E}_t \left[(P_{t+k} C_{t+k})^{-1} \tilde{Y}_{t+k} P_{H,t+k} MC_{t+k} \right]}{\sum_{k=0}^{\infty} (\beta\theta)^k \mathrm{E}_t \left[(P_{t+k} C_{t+k})^{-1} \tilde{Y}_{t+k} \right]}, \text{ (1-1-82)}$$

with $\tilde{Y}_{t+k} \equiv \left(\frac{\tilde{P}_{H,t}}{P_{H,t+k}}\right)^{-\varepsilon_p} Y_{t+k}$ and:

$$MC_t \equiv \frac{W_t}{(1-\tau_t) P_{H,t} A_t}. \text{ (1-1-83)}$$

1.1.11 Optimal Wage setting

Consider a household resetting its wage in period *t* to maximize:

$$\tilde{U}_H \equiv \mathrm{E}_t \left(\sum_{k=0}^{\infty} \beta^t \tilde{U}_{H,t+k} \right). \text{ (1-1-84)}$$

with:

$$\tilde{U}_{H,t+k} \equiv \left[\ln C_{t+k} - \frac{1}{1+\varphi} \int_0^1 N_{t+k|t}(j)^{1+\varphi} dj \right] Z_t. \text{ (1-1-85)}$$

The maximization of Eq.(1-1-84) is the subject to the sequence of labor demand schedules and a sequences of budget constraints of the form:

$$N_{t+k|t}(j)=\left(\frac{\tilde{W}_t(j)}{W_{t+k}}\right)^{-\varepsilon_W} N_{t+k}\,, \text{ (1-1-86)}$$

$$R_{t-1}B_{t-1}^{n}+R_{t-1}^{*}B_{F,t-1}^{n*}E_t+\int_0^1 \tilde{W}_t(j)N_{t+k|t}(j)dj+PR_{H,t}^{n}\geq P_tC_t+B_{H,t}^{n}+B_{F,t}^{n*}E_t\,. \text{ (1-1-87)}$$

We now make explicit that the households can pool labor income risk through government debt. Each household j reoptimizing the wage at a given time t will choose the same optimal wage. Because of this, we can abstract from index j on Eqs. (1-1-85), (1-1-86) and (1-1-87). Then these are given by:

$$\tilde{U}_{H,t+k}\equiv\left(\ln C_{t+k}-\frac{1}{1+\psi}N_{t+k|t}^{1+\varphi}\right)D_{H,t}\,, \text{ (1-1-88)}$$

$$N_{t+k|t}=\left(\frac{\tilde{W}_t}{W_{t+k}}\right)^{-\varepsilon_W} N_{t+k}\,, \text{ (1-1-89)}$$

$$D_t^{n}+\tilde{W}_tN_{t+k|t}+PR_t^{n}\geq P_tC_t+\mathrm{E}_t\left(Q_{t,t+1}D_{t+1}^{n}\right). \text{ (1-1-90)}$$

Because of Eqs.(1-1-88) and (1-1-90), the Lagrangean is given by:

$$L\equiv\sum_{k=0}^{\infty}(\beta\theta_W)^k\,\mathrm{E}_t\left[\ln C_{t+k}-\frac{1}{1+\varphi}N_{t+K|t}^{1+\varphi}+\lambda_{t+k|t}\left(\tilde{W}_tN_{t+k|t}-P_{t+k}C_{t+k}\right)\right],$$

which can be rewritten as:

$$\begin{aligned}L\equiv&\left(\ln C_t-\frac{1}{1+\varphi}N_{t|t}^{1+\varphi}\right)Z_t+\lambda_{t|t}\left(\tilde{W}_tN_{H,t|t}-P_tC_t\right)\\&+\beta\theta_W\mathrm{E}_t\left[\left(\ln C_{t+1}-\frac{1}{1+\varphi}N_{t+1|t}^{1+\varphi}\right)Z_{t+1}+\lambda_{t+1|t}\left(\tilde{W}_tN_{H,t+1|t}-P_{t+1}C_{t+1}\right)\right]+\cdots\end{aligned}$$

The FONC is given by:

$$\begin{aligned}&-N_{t|t}^{\varphi}Z_t\frac{\partial N_{t|t}}{\partial\tilde{W}_t}+\lambda_{t|t}\left(N_{t|t}+\tilde{W}_{H,t}\frac{\partial N_{t|t}}{\partial\tilde{W}_t}\right)\\&+\beta\theta_W\mathrm{E}_t\left[-N_{t+1|t}^{\varphi}Z_{t+1}\frac{\partial N_{t+1|t}}{\partial\tilde{W}_{H,t}}+\lambda_{t+1|t}\left(N_{t+1|t}+\tilde{W}_t\frac{\partial N_{t+1|t}}{\partial\tilde{W}_t}\right)\right]+\cdots=0\end{aligned}.$$

The compact form of the previous expression is given by:

$$\sum_{k=0}^{\infty}(\beta\theta_W)^k\left[-N_{t+k|t}^{\varphi}Z_{t+k}\frac{\partial N_{t+k|t}}{\partial\tilde{W}_t}+\lambda_{t+k|t}\left(N_{t+k|t}+\tilde{W}_t\frac{\partial N_{t+k|t}}{\partial\tilde{W}_t}\right)\right]=0\,. \text{ (1-1-91)}$$

Notice that $-N_{t+k|t}^{\varphi} \equiv \frac{\partial \tilde{U}_{t+k}}{\partial N_{t+k|t}}$.

Partial derivative of Eq.(1-1-89) is given by:

$$\begin{aligned}\frac{\partial N_{t+k|t}}{\partial \tilde{W}_t} &= -\varepsilon_W \left(\frac{\tilde{W}_t}{W_{t+k}}\right)^{-\varepsilon_W - 1} \frac{1}{W_{t+k}} N_{t+k} \\ &= -\varepsilon_W \left(\frac{\tilde{W}_t}{W_{t+k}}\right)^{-\varepsilon_W} N_{t+k} \frac{W_{t+k}}{\tilde{W}_{H,t}} \frac{1}{W_{t+k}}, \text{ (1-1-92)} \\ &= -\varepsilon_W N_{t+k|t} \frac{1}{\tilde{W}_t}\end{aligned}$$

where Eq.(1-1-89) is used to eliminate $\left(\frac{\tilde{W}_t}{W_{t+k}}\right)^{-\varepsilon_W} N_{t+k}$ in the second line.

Plugging Eq.(1-1-92) into Eq.(1-1-91) yields:

$$\sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left\{\begin{array}{l} \varepsilon_W N_{t+k|t}^{1+\varphi} Z_{t+k} \frac{1}{\tilde{W}_{H,t}} \\ + \lambda_{t+k|t} \left[N_{t+k|t} + (-\varepsilon_W) N_{t+k|t}\right] \end{array}\right\} = \sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left[\begin{array}{l} \varepsilon_W N_{t+k|t}^{1+\varphi} Z_{t+k} \frac{1}{\tilde{W}_{H,t}} \\ + \lambda_{t+k|t} (1 - \varepsilon_W) N_{t+k|t} \end{array}\right] = 0$$

Multiplying -1 on both sides of the previous expression yields:

$$\sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left[-\varepsilon_W N_{t+k|t}^{1+\varphi} Z_{t+k} \frac{1}{\tilde{W}_{H,t}} + \lambda_{t+k|t} (\varepsilon_W - 1) N_{t+k|t}\right] = 0$$

Plugging $\lambda_{t+k|t} = \frac{1}{P_{t+k} C_{t+k}}$ into the previous expression yields:

$$\sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left[-\varepsilon_W N_{t+k|t}^{1+\varphi} \frac{1}{\tilde{W}_t} + \frac{1}{P_{t+k}} \frac{N_{t+k|t}}{C_{t+k}} (\varepsilon_W - 1)\right] = 0.$$

Multiplying both sides on the previous expression by $\frac{\tilde{W}_t}{\varepsilon_W - 1}$ yields:

$$\sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left(-\frac{\varepsilon_W}{\varepsilon_W - 1} N_{t+k|t}^{1+\varphi} + \frac{\tilde{W}_t}{P_{t+k}} \frac{N_{t+k|t}}{C_{t+k}}\right) = 0. \text{ (1-1-93)}$$

The LHS of Eq.(1-1-93) can be rewritten as:

$$\begin{aligned}
&\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+k|t}^{1+\varphi}+\frac{\tilde{W}_t}{P_{t+k}}\frac{N_{t+k|t}}{C_{t+k}}\right)\\
&=-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t|t}^{1+\varphi}+\frac{\tilde{W}_t}{P_t}\frac{N_{t|t}}{C_t}+\beta\theta_W\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+1|t}^{1+\varphi}+\frac{\tilde{W}_t}{P_{t+1}}\frac{N_{t+1|t}}{C_{t+1}}\right)\\
&+(\beta\theta_W)^2\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+2|t}^{1+\varphi}+\frac{\tilde{W}_t}{P_{t+2}}\frac{N_{t+2|t}}{C_{t+2}}\right)+\cdots\\
&=-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t|t}^{\varphi}C_t\frac{N_{t|t}}{C_t}+\frac{\tilde{W}_t}{P_t}\frac{N_{H,t|t}}{C_t}+\beta\theta_W\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+1|t}^{\varphi}C_{t+1}\frac{N_{t+1|t}}{C_{t+1}}+\frac{\tilde{W}_t}{P_{t+1}}\frac{N_{t+1|t}}{C_{t+1}}\right)\\
&+(\beta\theta_W)^2\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+2|t}^{1+\varphi}C_{t+2}\frac{N_{t+2|t}}{C_{t+2}}+\frac{\tilde{W}_t}{P_{t+2}}\frac{N_{t+2|t}}{C_{t+2}}\right)+\cdots\\
&=-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t|t}^{\varphi}C_t\frac{N_{t|t}}{C_t}+\frac{\tilde{W}_t}{P_t}\frac{N_{t|t}}{C_t}+\beta\theta_W\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+1|t}^{\varphi}C_{t+1}\frac{N_{t+1|t}}{C_{t+1}}+\frac{\tilde{W}_t}{P_{t+1}}\frac{N_{t+1|t}}{C_{t+1}}\right)\\
&+(\beta\theta_W)^2\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}N_{t+2|t}^{1+\varphi}C_{t+2}\frac{N_{t+2|t}}{C_{t+2}}+\frac{\tilde{W}_t}{P_{t+2}}\frac{N_{t+2|t}}{C_{t+2}}\right)+\cdots\\
&=-\frac{\varepsilon_W}{\varepsilon_W-1}MRS_{t|t}\frac{N_{t|t}}{C_t}+\frac{\tilde{W}_t}{P_t}\frac{N_{t|t}}{C_t}+\beta\theta_W\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}MRS_{t+1|t}\frac{N_{t+1|t}}{C_{t+1}}+\frac{\tilde{W}_t}{P_{t+1}}\frac{N_{t+1|t}}{C_{t+1}}\right).\\
&+(\beta\theta_W)^2\mathrm{E}_t\left(-\frac{\varepsilon_W}{\varepsilon_W-1}MRS_{t+2|t}\frac{N_{t+2|t}}{C_{t+2}}+\frac{\tilde{W}_t}{P_{t+2}}\frac{N_{t+2|t}}{C_{t+2}}\right)+\cdots=0
\end{aligned}$$

Note that:

$$MRS_{t+k|t}\equiv-\frac{\partial U_{t+k|t}/\partial N_{t+k|t}}{\partial U_{t+k}/\partial C_{t+k}}=\frac{N_{t+k|t}^{\varphi}}{C_{t+k}^{-1}}. \text{ (1-1-93)'}$$

Further, the last equality can be rewritten as:

$$\frac{\varepsilon_W}{\varepsilon_W-1}\begin{bmatrix}MRS_{t|t}\dfrac{N_{t|t}}{C_t}+\beta\theta_W\mathrm{E}_t\left(MRS_{H,t+1|t}\dfrac{N_{t+1|t}}{C_{t+1}}\right)\\+(\beta\theta_W)^2\mathrm{E}_t\left(MRS_{t+2|t}\dfrac{N_{t+2|t}}{C_{t+2}}\right)\cdots\end{bmatrix}=\tilde{W}_t\begin{bmatrix}\dfrac{N_{t|t}}{P_tC_t}+\beta\theta_W\mathrm{E}_t\left(\dfrac{N_{t+1|t}}{P_{t+1}C_{t+1}}\right)\\+(\beta\theta_W)^2\mathrm{E}_t\left(\dfrac{N_{t+2|t}}{P_{t+2}C_{t+2}}\right)\cdots\end{bmatrix},$$

which can be a compact form as follows:

$$\frac{\varepsilon_W}{\varepsilon_W-1}\sum_{k=0}^{\infty}(\beta\theta_W)^k\mathrm{E}_t\left(MRS_{t+k|t}N_{t+k|t}C_{t+k}^{-1}\right)=\tilde{W}_t\sum_{k=0}^{\infty}(\beta\theta_W)^k\mathrm{E}_t\left[N_{t+k|t}\left(P_{t+k}C_{t+k}\right)^{-1}\right]. \text{ (1-1-94)}$$

Eq.(1-1-94) can be rewritten as:

$$\tilde{W}_t = \frac{\varepsilon_W/(\varepsilon_W - 1)\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left(MRS_{t+k|t}N_{t+k|t}C_{t+k}^{-1}\right)}{\sum_{k=0}^{\infty}(\beta\theta_W)^k \mathrm{E}_t\left[N_{t+k|t}\left(P_{t+k}C_{t+k}\right)^{-1}\right]}.$$

(1-1-95)

Given the assumed wage structure, the evolution of the aggregate wage index is given by:

$W_t = \left[\theta_W W_{t-1}^{1-\varepsilon_W} + (1-\theta_W)\tilde{W}_t^{1-\varepsilon_W}\right]^{\frac{1}{1-\varepsilon_W}}$. (1-1-97)

4 Nonstochastic Steady State

We focus on equilibria where the state variables follow paths that are close to a deterministic stationary equilibrium, in which $\Pi_{H,t} = \Pi_t = 1$, $S = 1$ and $\beta = R^{-1} = \left(R^*\right)^{-1}$. Because this steady state is nonstochastic, the productivity has unit values; i.e., $A = 1$.

2.2 Steady State Relative Price and Market Clearing

In the steady state, we assume $P_H = P_F$ and then $S = 1$ is applied. Because of $S = 1$. Plugging Eq.(4-6-3) into Eq.(4-14-17) yields:

$P = P^*$,

which implies that the PPP is applicable in the steady state. Plugging the previous expression into the definition of the CPI disparity yields:

$Q = 1$. (2-7)

Plugging Eq.(2-7) into the international risk sharing condition $C = QC^*\vartheta$, we have:

$C = C^*$ (2-8),

where we impose $\vartheta = 1$.

2.3 Steady State Fiscal Surplus and Government Debt

The definition of the fiscal surplus implies:

$SP = \tau Y - G$. (2-11).

Eq.(1-1-45) implies as:

$B_t = R_{t-1}B_{t-1}\Pi_t^{-1} - SP_t$.

Then, in the steady state:

$$SP=(R-1)B$$
$$=\left(\frac{1-\beta}{\beta}\right)B \text{ (2-24).}$$

2.4 Steady State Wedge between Marginal Utility of Consumption and Labor

The FONC for households to choose optimal wage Eq.(1-1-93) implies that:

$$-\frac{\varepsilon_W}{\varepsilon_W-1}N^{1+\varphi}+\frac{W}{P}\frac{N}{C}+\beta\theta_W\left[-\frac{\varepsilon_W}{\varepsilon_W-1}N^{1+\varphi}+\frac{W}{P}\frac{N}{C}\right]$$
$$+(\beta\theta_W)^2\left[-\frac{\varepsilon_W}{\varepsilon_W-1}N^{1+\varphi}+\frac{W}{P}\frac{N}{C}\right]+\cdots=0,$$

which can be rewritten as:

$$\left[1+\beta\theta_W(\beta\theta_W)^2+\cdots\right]\frac{W}{P}\frac{N}{C}=\left[1+\beta\theta_W(\beta\theta_W)^2+\cdots\right]\frac{\varepsilon_W}{\varepsilon_W-1}N^{1+\varphi}.$$

Then, we have:

$$\frac{W}{P}=\frac{\varepsilon_W}{\varepsilon_W-1}N^{\varphi}C. \text{ (2-27)}$$

Eq. (1-1-83)implies as follows:

$$MC=\frac{1}{1-\tau}\frac{W}{P}. \text{ (2-27)'}$$

Combining Eqs.(2-27) and (2-27)' yields:

$$MC(1-\tau)=\frac{\varepsilon_W}{\varepsilon_W-1}N^{\varphi}C. \text{ (2-27)''}$$

The FONC for the firms Eq.(1-1-82) implies that:

$$1=\frac{\varepsilon_p}{\varepsilon_p-1}MC,$$

which can be rewritten as:

$$MC=\frac{\varepsilon_p-1}{\varepsilon_p}. \quad \text{(2-28)}$$

The definition of the marginal cost $MC_t\equiv\frac{W_t}{P_{H,t}A_t}$ implies that:

$$MC=\frac{W}{P}. \text{ (2-31)}$$

Because of Eqs(2-30), (2-32) and (2-32)', Eq.(2-27)'' can be rewritten as:

$$\frac{\varepsilon_W}{\varepsilon_W - 1} N^{\varphi} C = \frac{(\varepsilon_p - 1)(1-\tau)}{\varepsilon_p} . \text{ (2-33)}$$

Note that $U_C = C^{-1}$ and $-U_N = N^{\varphi}$. Plugging these conditions into Eq.(2-33) yields:

$$-\frac{U_N}{U_C} = \frac{1-\tau}{MM^W} . \text{ (2-34)}$$

with $M^p \equiv \frac{\varepsilon_p}{\varepsilon_p - 1}$ and $M^W \equiv \frac{\varepsilon_W}{\varepsilon_W - 1}$.

Plugging $-\frac{U_N}{U_C} \equiv 1 - \Phi$ into the previous expression, we have:

$$1 - \Phi = \frac{1-\tau}{M^p M^W} \quad \text{or} \quad \Phi = 1 - \frac{1-\tau}{M^p M^W} , \text{ (2-35)}$$

which shows the steady state wedge between marginal utility of consumption and its labor.

5 Log-linearization of the Model

3.1 FTPL Model

3.1.1 Relationship between Real Exchange Rate and Terms of Trade

Note that:

$$\begin{aligned} q_t &= e_t + p_t^* - p_t \\ &= e_t + p_{F,t}^* - p_t \\ &= p_{F,t} - (1-\upsilon) p_{H,t} - \alpha p_{F,t} . \text{ (3-1-4)} \\ &= (1-\upsilon)(p_{F,t} - p_{H,t}) \\ &= (1-\upsilon) s_t \end{aligned}$$

3.1.2 Market Clearing Condition

By Using $X_{H,t} \equiv P_{H,t} / P_t$, Eq.(1-1-76) can be rewritten as:

$$Y_t = (1-\upsilon) X_{H,t}^{-\eta} C_t + \upsilon S_t^{\eta} Z_{1,t}^* + G_t ,$$

Where we use Eq.(1-1-20).

Total derivative of the previous expression is given by:

$$dY_t = (1-\upsilon)(-\eta)CdX_{H,t} + (1-\upsilon)dC_t + \upsilon\eta CdS_t + \upsilon dZ^*_{1,t} + dG_t .$$

Dividing both sides by Y yields:

$$\frac{dY_t}{Y} = -\eta(1-\upsilon)\frac{C}{Y}dX_{H,t} + (1-\upsilon)\frac{C}{Y}\frac{dC_t}{C} + \upsilon\eta\frac{C}{Y}dS_t + \upsilon\frac{C}{Y}\frac{Z^*_1}{C^*}\frac{dZ^*_{1,t}}{Z^*_1} + \frac{G}{Y}\frac{dG_t}{G},$$

which can be rewritten as:

$$y_t = -\eta(1-\upsilon)\sigma_C x_t + \eta\upsilon\sigma_C s_t + (1-\upsilon)\sigma_C c_t + \upsilon\sigma_C z^*_{1,t} + \sigma_G g_t \quad \text{(3-1-6)}$$

with $\sigma_C \equiv C/Y$ and $\sigma_G \equiv G/Y = 1-\sigma_C$. Note that:

$$\begin{aligned} x_{H,t} &= p_{H,t} - p_t \\ &= p_{H,t} - (1-\upsilon)p_{H,t} - \upsilon p_{F,t} \\ &= -\upsilon(p_{F,t} - p_{H,t}) \\ &= -\upsilon s_t \end{aligned} \quad \text{(3-1-7)}$$

Plugging the last line in Eq.(3-1-7) into Eq(3-1-6) yields:

$$\begin{aligned} y_t &= -\eta(1-\upsilon)\sigma_C(-\upsilon s_t) + \eta\upsilon\sigma_C s_t + (1-\upsilon)\sigma_C c_t + \upsilon\sigma_C z^*_{1,t} + \sigma_G g_t \\ &= \eta\sigma_C\upsilon(2-\upsilon)s_t + (1-\upsilon)\sigma_C c_t + \upsilon\sigma_C z^*_{1,t} + \sigma_G g_t \end{aligned}$$ (3-1-8)' [(43) in the text]

3.1.3 International Risk Sharing Condition

Total derivative of Eq.(1-1-21) is given by:

$$\begin{aligned} dC_t &= dQ_t + \frac{1}{Z^*_2}dZ_t - Z(Z^*_2)^{-2}dZ^*_{2,t} \\ &= \frac{Z}{Z^*_2}dQ_t + \frac{Z}{Z^*_2}\frac{dZ_t}{Z} - \frac{Z}{Z^*_2}\frac{dZ^*_{2,t}}{Z^*_2} \end{aligned}$$

Dividing both sides by C yields:

$$\begin{aligned} \frac{dC_t}{C} &= \frac{Z^*_2}{Z}\left(\frac{Z}{Z^*_2}dQ_t + \frac{Z}{Z^*_2}\frac{dZ_t}{Z} - \frac{Z}{Z^*_2}\frac{dZ^*_{2,t}}{Z^*_2}\right) \\ &= dQ_t + \frac{dZ_t}{Z} - \frac{dZ^*_{2,t}}{Z^*_2} \end{aligned},$$

which can be rewritten as:

$$c_t = q_t + z_t - z^*_{2,t} \quad \text{(3-1-10)}$$

Plugging the last line of Eq.(3-1-4) into Eq.(3-1-10) yields:

$c_t = (1-\upsilon)s_t + z_t - z_{2,t}^*$ (3-1-11)

Plugging Eq.(3-1-11) into Eq.(3-1-8)' yields:

$$\begin{aligned} y_t &= \eta\upsilon(2-\upsilon)s_t + (1-\upsilon)\left[(1-\upsilon)s_t + z_t - z_{2,t}^*\right] + \upsilon z_{1,t}^* \\ &= \left[\eta\upsilon(2-\upsilon) + (1-\upsilon)^2\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \\ &= \left(2\eta\upsilon - \eta\upsilon^2 + 1 - 2\upsilon + \upsilon^2\right)s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \\ &= \left[2\upsilon(\eta-1) - \upsilon^2(\eta-1) + 1\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \\ &= \left[(\eta-1)(2\upsilon-\upsilon^2) + 1\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \\ &= \left[(\eta-1)\upsilon(2-\upsilon) + 1\right]s_t + (1-\upsilon)z_t + \upsilon z_{1,t}^* - (1-\upsilon)z_{2,t}^* \end{aligned} \quad . \text{(3-1-12)}$$

3.1.4 Iterated Government Budget Constraint with Euler Equation

Iterated Government Budget Constraint with Euler Equation Eq.(1-1-67) can be rewritten as:

$SP_t = B_{t-1}\Pi_t^{-1} - \beta E_t\left(C_{t+1}^{-1}C_t Z_{t+1} Z_t^{-1} B_t \Pi_{t+1}^{-1}\right)$. (1-1-67)

Total derivative of the previous expression is given by:

$$\begin{aligned} dSP_{H,t} &= dB_{t-1} - Bd\Pi_t \\ &\quad -\beta\left[CB(-1)C^{-2}dC_{t+1} + C^{-1}BdC_t + dB_t + B(-1)d\Pi_{t+1} + B(-1)dZ_t + BdZ_{t+1}\right] \\ &= dB_{t-1} - Bd\Pi_t + \beta B\frac{dC_{t+1}}{C} - \beta B\frac{dC_t}{C} - \beta dB_t + \beta Bd\Pi_{t+1} + \beta BdZ_t \\ &\quad -\beta BdZ_{t+1} \end{aligned}$$

Dividing both sides on the previous expression by SP_t yields:

$$\begin{aligned} \frac{dSP_t}{SP} &= \frac{1}{(1-\beta)B}\left[\begin{aligned} &dB_{t-1} - Bd\Pi_t + \beta B\frac{dC_{t+1}}{C} - \beta B\frac{dC_t}{C} - \beta dB_t + \beta Bd\Pi_{t+1} + \beta BdZ_t \\ &-\beta BdZ_{t+1} \end{aligned}\right] \\ &= \frac{1}{1-\beta}\frac{dB_{t-1}}{B} - \frac{1}{1-\beta}d\Pi_t + \frac{\beta}{1-\beta}\frac{dC_{t+1}}{C} - \frac{\beta}{1-\beta}\frac{dC_t}{C} \\ &\quad - \frac{\beta}{1-\beta}\frac{dB_t}{B} + \frac{\beta}{1-\beta}d\Pi_{t+1} + \frac{\beta}{1-\beta}dZ_t - \frac{\beta}{1-\beta}dZ_{t+1} \end{aligned} ,$$

where we use Eq.(2-24).

The previous expression can be rewritten as:

$$sp_t = \frac{1}{1-\beta}b_{t-1} - \frac{1}{1-\beta}\pi_t + \frac{\beta}{1-\beta}\mathrm{E}_t\left(c_{t+1}\right) - \frac{\beta}{1-\beta}c_t - \frac{\beta}{1-\beta}b_t + \frac{\beta}{1-\beta}\mathrm{E}_t\left(\pi_{t+1}\right) + \frac{\beta}{1-\beta}z_t$$
$$-\frac{\beta}{1-\beta}\mathrm{E}_t\left(z_{t+1}\right)$$
$$= \frac{1}{1-\beta}b_{t-1} - \frac{1}{1-\beta}\pi_t + \frac{\beta}{1-\beta}\mathrm{E}_t\left(c_{t+1}\right) - \frac{\beta}{1-\beta}c_t - \frac{\beta}{1-\beta}b_t + \frac{\beta}{1-\beta}\mathrm{E}_t\left(\pi_{t+1}\right) + \frac{\beta\left(1-\rho_z\right)}{1-\beta}z_t$$

.

Rearranging the previous expression yields:

$c_t = \mathrm{E}_t\left(c_{t+1}\right) + \mathrm{E}_t\left(\pi_{t+1}\right) - b_t - \frac{1}{\beta}\pi_t + \frac{1}{\beta}b_{t-1} - \frac{1-\beta}{\beta}sp_t + \left(1-\rho_z\right)z_t$, (3-1-16) [(44) in the text]

with $\delta \equiv \beta^{-1} - 1 = r$ being time discount rate where we use the fact that $\mathrm{E}_t\left(z_{t+1}\right) = \rho_z z_t$.

3.1.5 Price Index and the Definition of the TOT

The definition of the TOT is given by $S_t \equiv P_{F,t}/P_{H,t}$, which can be log-linearized as:

$$s_t = p_{F,t} - p_{H,t}$$
$$= e_t + p_{F,t}^* - p_{H,t} \text{ . (3-1-17)'}$$
$$= e_t - p_{H,t}$$

Log-linearizing Eq.(1-1-78) yields:

$$p_t = \left(1-\upsilon\right)p_{H,t} + \upsilon p_{F,t}$$
$$= p_{H,t} + \upsilon\left(p_{F,t} - p_{H,t}\right) \text{. (3-1-17)}$$
$$= p_{H,t} + \upsilon s_t$$

Combining the previous expression and its lagged expression yields:

$$\pi_t = \left(1-\upsilon\right)\pi_{H,t} + \upsilon\pi_{F,t},$$

Where:

$\pi_{H,t} \equiv p_{H,t} - p_{H,t-1}$. (3-1-17)''

because of inflation targeting in the ROW, $p_{F,t}^* = 0$ for all t.

Combining the previous expression and its lagged expression yields:

$s_t - s_{t-1} = \pi_{F,t} - \pi_{H,t}$ (3-1-18)

Plugging Eq(3-1-18) into eq.(3-1-17) yields:

$$\begin{aligned}\pi_t &= (1-\upsilon)\pi_{H,t} + \upsilon(\pi_{H,t} + s_t - s_{t-1}) \\ &= \pi_{H,t} + \upsilon s_t - \upsilon s_{t-1}\end{aligned}.$$

The definition of the CPI $\Pi_t \equiv \dfrac{P_t}{P_{t-1}}$ can be log-linearized as:

$\pi_t \equiv p_t - p_{t-1}$. (3-1-19)

3.1.6 Flow Government Budget Constraint and the Definition of the Fiscal Surplus

Eq.(1-1-45) can be rewritten as:

$SP_t = R_{t-1}B_{t-1}\Pi_t^{-1} - B_t$, (1-1-45)

Total derivative of the previous expression is given by:

$$dSP_t = -dB_t + BdR_{t-1} + RdB_{t-1} - RBd\Pi_t$$

Dividing both sides on the previous expression by $SP = \left(\dfrac{1-\beta}{\beta}\right)B$ yields:

$$\begin{aligned}\frac{dSP_t}{SP} &= -\frac{\beta}{1-\beta}\frac{dB_t}{B} + \frac{1}{1-\beta}\frac{dR_{t-1}}{R} + \frac{1}{1-\beta}\frac{dB_{t-1}}{B} - \frac{1}{1-\beta}d\Pi_t \\ &= -\frac{\beta}{1-\beta}\frac{dB_t}{B} + \frac{1}{1-\beta}\ln\left(\frac{1+r_{t-1}}{1+r}\right) + \frac{1}{1-\beta}\frac{dB_{t-1}}{B} - \frac{1}{1-\beta}d\Pi_t\end{aligned},$$

which can be rewritten as:

$$sp_t = -\frac{\beta}{1-\beta}b_t + \frac{1}{1-\beta}r_{t-1} + \frac{1}{1-\beta}b_{t-1} - \frac{1}{1-\beta}\pi_t - \frac{1}{1-\beta}\delta.$$

Further:

$b_t = \dfrac{1}{\beta}r_{t-1} - \dfrac{1}{\beta}\pi_t + \dfrac{1}{\beta}b_{t-1} - \dfrac{1-\beta}{\beta}sp_t - \dfrac{1}{\beta}\delta$. (3-1-21) [(46) in the text]

Total derivative of Eq(1-1-46) is given by:

$$dSP_t = SPdX_t + Yd\tau_t + \tau dY_t - dG_t - d\zeta_t$$

Dividing both sides on the previous expression by SP_H yields:

$$\begin{aligned}\frac{dSP_t}{SP} &= \frac{\beta}{1-\beta}\frac{1}{B}(SPdX_t + Yd\tau_t + \tau dY_t - dG_t - d\zeta_t) \\ &= dX_t + \frac{\beta}{1-\beta}\frac{Y}{B}d\tau_t + \frac{\beta}{1-\beta}\frac{Y}{B}\tau\frac{dY_t}{Y} - \frac{\beta}{1-\beta}\frac{Y}{B}\frac{G}{Y}\frac{dG_t}{G} - \frac{\beta}{1-\beta}\frac{Y}{B}\frac{d\zeta_t}{Y} \\ &= dX_t + \frac{\beta}{1-\beta}\left(\frac{B}{Y}\right)^{-1}d\tau_t + \frac{\beta\tau}{1-\beta}\left(\frac{B}{Y}\right)^{-1}\frac{dY_t}{Y} - \frac{\beta}{1-\beta}\left(\frac{B}{Y}\right)^{-1}\frac{G}{Y}\frac{dG_t}{G} \\ &\quad -\frac{\beta}{1-\beta}\left(\frac{B}{Y}\right)^{-1}\frac{d\zeta_t}{Y}\end{aligned}.$$

Note that $\tau_t = \left(\frac{d\tau_t}{\tau} + 1\right)\tau$ which can be rewritten as $d\tau_t = \tau_t - \tau$. Thus, the previous expression can be rewritten as:

$$sp_t = x_{H,t} + \frac{\beta}{(1-\beta)\sigma_B}\tau_t + \frac{\beta\tau}{(1-\beta)\sigma_B}y_t - \frac{\beta\sigma_G}{(1-\beta)\sigma_B}g_t - \frac{\beta}{(1-\beta)\sigma_B}\hat{\zeta}_t - \frac{\beta}{(1-\beta)\sigma_B}\tau,$$

with $\hat{\zeta}_t \equiv \frac{d\zeta_t}{Y}$, where we use Eq.(2-23). Plugging Eq(1-1-50) into the previous expression yields:

$$sp_t = -\upsilon s_t + \frac{\beta}{(1-\beta)\sigma_B}\tau_t + \frac{\beta\tau}{(1-\beta)\sigma_B}y_t - \frac{\beta\sigma_G}{(1-\beta)\sigma_B}g_t - \frac{\beta}{(1-\beta)\sigma_B}\hat{\zeta}_t - \frac{\beta}{(1-\beta)\sigma_B}\tau$$

. (3-1-22) [(45) in the text]

3.1.7 NKPC

The NKPC is given by:

$$\pi_{H,t} = \beta \mathrm{E}_t\left(\pi_{H,t+1}\right) + \kappa mc_t, \text{ (3-1-23)}$$

with $\kappa_p \equiv \frac{\left(1-\theta_p\right)\left(1-\beta\theta_p\right)}{\theta_p}$.

Total derivative of Eq.(1-1-83) yields:

$$\begin{aligned} dMC_t &= \left(\frac{W}{P}\right)(-1)(1-\tau)^{-2}(-1)d\tau_t + \frac{1}{1-\tau}d\left(\frac{W_t}{P_{H,t}}\right) + \frac{1}{1-\tau}\left(\frac{W}{P}\right)(-1)dA_t \\ &= \frac{1}{1-\tau}\left(\frac{W}{P}\right)\frac{\tau}{1-\tau}\frac{d\tau_t}{\tau} + \frac{1}{1-\tau}d\left(\frac{W_t}{P_{H,t}}\right) - \frac{1}{1-\tau}\left(\frac{W}{P}\right)dA_t \end{aligned}.$$

Dividing both sides of the previous expression by $MC = \frac{1}{1-\tau}\frac{W}{P}$ yields:

$$\frac{dMC_t}{MC} = \frac{1}{1-\tau}d\tau_t + \frac{d\left(W_{H,t}/P_{H,t}\right)}{(W/P)} - dA_{H,t}.$$

As mentioned, $d\tau_t = \tau_t - \tau$. Thus, we have:

$$mc_t = \frac{1}{1-\tau}\tau_t + w_t^r - a_t - \frac{1}{1-\tau}\tau, \text{ (3-1-24)}$$

with $w_{H,t}^{r} \equiv \frac{d\left(W_{H,t}/P_{H,t}\right)}{\left(W/P\right)}$ being the (log) real wage.

Eq.(1-1-80) can be rewritten as:

$N_t = \frac{Y_t \Delta_{p,t}}{A_t}$.

Total differential of the previous expression yields:

$$\begin{aligned} dN_t &= dY_t + (-1)dA_t + Yd\Delta_{p,t} \\ &= dY_t - dA_t + d\Delta_{p,t} \end{aligned},$$

where $\ln\Delta_{p,t}$ is $o\left(\|\xi\|^2\right)$ and is omitted.

$n_t = y_t - a_t$. (3-1-26)

where $\ln\Delta_{p,t}$ is $o\left(\|\xi\|^2\right)$ and is omitted.

3.1.8 Wage Inflation Dynamics

Total derivative of Eq.(1-1-95) is given by:

$$\begin{aligned} d\tilde{W}_t = {} & \mathcal{M}^w \begin{Bmatrix} (-1)C^{-2}NMRSdC_t + C^{-1}MRSd\mathcal{N}_t + C^{-1}Nd\mathcal{MRS}_t \\ +\beta\theta_w\left[(-1)C^{-2}NMRSdC_{t+1} + C^{-1}MRSd\mathcal{N}_{t+1|t} + C^{-1}Nd\mathcal{MRS}_{t+1|t}\right] \\ +(\beta\theta_w)^2\left[(-1)C^{-2}NMRSdC_{t+1} + C^{-1}MRSd\mathcal{N}_{t+1|t} + C^{-1}Nd\mathcal{MRS}_{t+1|t}\right] \\ +\cdots \end{Bmatrix} \\ & \times\left\{C^{-1}NP^{-1}\left[1+\beta\theta_w + (\beta\theta_w)^2 + \cdots\right]\right\}^{-1} \\ & +(-1)\left\{C^{-1}NP^{-1}\left[1+\beta\theta_w + (\beta\theta_w)^2 + \cdots\right]\right\}^{-2} \\ & \times\begin{Bmatrix} (-1)C^{-2}NP^{-1}dC_t + C^{-1}P^{-1}d\mathcal{N}_t + (-1)C^{-1}NP^{-2}dP_t \\ +\beta\theta_w\left[(-1)C^{-2}NP^{-1}dC_{t+1} + C^{-1}P^{-1}d\mathcal{N}_{t+1|t} + (-1)C^{-1}NP^{-2}dP_{t+1}\right] \\ +(\beta\theta_w)^2\left[(-1)C^{-2}NP^{-1}dC_{t+2} + C^{-1}P^{-1}d\mathcal{N}_{t+2|t} + (-1)C^{-1}NP^{-2}dP_{t+2}\right] \\ +\cdots \end{Bmatrix} \\ & \times\mathcal{M}^w C^{-1}NMRS\left[1+\beta\theta_w + (\beta\theta_w)^2 + \cdots\right] \end{aligned}.$$

Then:

$$d\tilde{W}_t = M^w MRSC^{-1}N \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} + \frac{dMRS_t}{MRS} \\ +\beta\theta_w \left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} + \frac{dMRS_{t+1|t}}{MRS} \right) \\ (\beta\theta_w)^2 \left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} + \frac{dMRS_{t+2|t}}{MRS} \right) + \cdots \end{bmatrix} CN^{-1}P \left(\frac{1}{1-\beta\theta_w} \right)^{-1}$$

$$-M^w MRSCN^{-1}P^2C^{-1}NP^{-1} \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} - \frac{dP_t}{P} \\ +\beta\theta_w \left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} - \frac{dP_{t+1}}{P} \right) \\ (\beta\theta_w)^2 \left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} - \frac{dP_{t+2}}{P} \right) + \cdots \end{bmatrix} \left(\frac{1}{1-\beta\theta_w} \right)^{-1}$$

$$= M^w PMRS \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} + \frac{dMRS_t}{MRS} \\ +\beta\theta_w \left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} + \frac{dMRS_{t+1|t}}{MRS} \right) \\ (\beta\theta_w)^2 \left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} + \frac{dMRS_{t+2|t}}{MRS} \right) + \cdots \end{bmatrix} \left(\frac{1}{1-\beta\theta_w} \right)^{-1}$$

$$-M^w PMRS \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} - \frac{dP_t}{P} \\ +\beta\theta_w \left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} - \frac{dP_{t+1}}{P} \right) \\ (\beta\theta_w)^2 \left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} - \frac{dP_{t+2}}{P} \right) + \cdots \end{bmatrix} \left(\frac{1}{1-\beta\theta_w} \right)^{-1} .$$

Note that $\tilde{W} = PM^w MRS$. Thus:

$$\frac{d\tilde{W}_t}{\tilde{W}} = \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} + \frac{dMRS_t}{MRS} \\ +\beta\theta_w\left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} + \frac{dMRS_{t+1|t}}{MRS}\right) \\ \left(\beta\theta_w\right)^2\left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} + \frac{dMRS_{t+2|t}}{MRS}\right) + \cdots \end{bmatrix} \left(\frac{1}{1-\beta\theta_w}\right)^{-1}$$

$$- \begin{bmatrix} -\frac{dC_t}{C} + \frac{dN_t}{N} - \frac{dP_t}{P} \\ +\beta\theta_w\left(-\frac{dC_{t+1}}{C} + \frac{dN_{t+1|t}}{N} - \frac{dP_{t+1}}{P}\right) \\ +\left(\beta\theta_w\right)^2\left(-\frac{dC_{t+2}}{C} + \frac{dN_{t+2|t}}{N} - \frac{dP_{t+2}}{P}\right) + \cdots \end{bmatrix} \left(\frac{1}{1-\beta\theta_w}\right)^{-1} ,$$

$$= \left(1-\beta\theta_w\right) \begin{bmatrix} \frac{dMRS_t}{MRS} + \beta\theta_w \frac{dMRS_{t+1|t}}{MRS} + \left(\beta\theta_w\right)^2 \frac{dMRS_{t+2|t}}{MRS} \\ + \cdots + \frac{dC_t}{C} + \beta\theta_w \frac{dC_{t+1}}{C} + \left(\beta\theta_w\right)^2 \frac{dC_{t+2}}{C} + \cdots \end{bmatrix}$$

which can be rewritten as:

$$\tilde{w}_t = \left(1-\beta\theta_w\right)\sum_{k=0}^{\infty}\left(\beta\theta_w\right)^k \mathrm{E}_t\left(mrs_{t+k|t} + p_{t+k}\right). \text{ (3-1-27)}$$

Total derivative of Eq.(1-1-93)' is given by:

$$\begin{aligned} dMRS_{t+k|t} &= C\varphi N^{\varphi-1} dN_{t+k|t} + N^{\varphi} dC_{t+k} \\ &= CN^{\varphi}\varphi \frac{dN_{t+k|t}}{N} + CN^{\varphi}\frac{dC_{t+k}}{C} . \end{aligned}$$

Note that $MRS = CN^{\varphi}$. Dividing both sides of the previous expression by MRS_H yields:

$$\frac{dMRS_{t+k|t}}{MRS} = \varphi\frac{dN_{t+k|t}}{N} + \frac{dC_{t+k}}{C} ,$$

which can be rewritten as:

$$mrs_{t+k|t} = \varphi n_{t+k|t} + c_{t+k} . \text{ (3-1-28)}$$

Let define $MRS_{H,t+k} \equiv \frac{N_{H,t+k}^{\varphi}}{C_{t+k}}$. Total derivative of this definition is given by:

$$dMRS_{t+k} = C\varphi N^{\varphi-1}dN_{t+k} + N^{\varphi}dC_{t+k}$$
$$= CN^{\varphi}\varphi\frac{dN_{t+k}}{N} + CN^{\varphi}\frac{dC_{t+k}}{C},$$

Dividing both sides of the previous expression yields:

$$\frac{dMRS_{t+k}}{MRS} = \varphi\frac{dN_{t+k}}{N} + \frac{dC_{t+k}}{C},$$

because of $MRS_H = CN^{\varphi}$.

That expression can be rewritten as:

$mrs_{t+k} = \varphi n_{t+k} + c_{t+k}$. (3-1-29)

Subtracting Eq.(3-1-29) from Eq.(3-1-28) yields:

$mrs_{t+k|t} = mrs_{t+k} + \varphi\left(n_{t+k|t} - n_{t+k}\right)$. (3-1-30)

Total derivative of Eq.(1-1-89) is given by:

$$dN_{t+k|t} = -\varepsilon_W W^{-\varepsilon_W-1}W^{\varepsilon_W}Nd\tilde{W}_t + \varepsilon_W W^{\varepsilon_W-1}W^{-\varepsilon_W}NdW_{t+k} + dN_{t+k}$$
$$= -\varepsilon_W N\frac{d\tilde{W}_t}{\tilde{W}} + \varepsilon_W N\frac{dW_{t+k}}{W} + dN_{t+k}.$$

Dividing both sides of previous expression by $N = N$ yields:

$$\frac{dN_{t+k|t}}{N} = -\varepsilon_W\frac{d\tilde{W}_t}{\tilde{W}} + \varepsilon_W\frac{dW_{t+k}}{W} + \frac{dN_{t+k}}{N},$$

Which can be rewritten as:

$n_{t+k|t} = -\varepsilon_W\left(\tilde{w}_t - w_{t+k}\right) + n_{t+k}$. (3-1-31)

Plugging Eq.(3-1-31) into Eq.(3-1-30) yields:

$mrs_{t+k|t} = mrs_{t+k} - \varepsilon_w\varphi\left(\tilde{w}_t - w_{t+k}\right)$. (3-1-32)

Plugging Eq.(3-1-32) into Eq.(3-1-27) yields:

$$
\begin{aligned}
\tilde{w}_{H,t} &= (1-\beta\theta_w)\sum_{k=0}^{\infty}(\beta\theta_w)^k \mathrm{E}_t\left[mrs_{t+k}-\varepsilon_w\varphi(\tilde{w}_t-w_{t+k})+p_{t+k}\right] \\
&= (1-\beta\theta_w)\begin{Bmatrix} mrs_t-\varepsilon_w\varphi(\tilde{w}_t-w_t)+p_t+\beta\theta_w\left[mrs_{t+1}-\varepsilon_w\varphi(\tilde{w}_t-w_{t+1})+p_{t+1}\right] \\ +(\beta\theta_w)^2\left[mrs_{t+2}-\varepsilon_w\varphi(\tilde{w}_t-w_{t+2})+p_{t+2}\right]+\cdots \end{Bmatrix} \\
&= (1-\beta\theta_w)\begin{Bmatrix} mrs_t+\varepsilon_w\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_w)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \\ -\varepsilon_w\varphi\tilde{w}_t-\varepsilon_w\varphi\beta\theta_w\tilde{w}_t-(\beta\theta_w)^2\varepsilon_w\varphi\tilde{w}_t \end{Bmatrix} \\
&= (1-\beta\theta_w)\begin{Bmatrix} mrs_t+\varepsilon_w\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_w)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \\ -\varepsilon_w\varphi\tilde{w}_t\left[1+\beta\theta_w+(\beta\theta_w)^2+\cdots\right] \end{Bmatrix} \\
&= (1-\beta\theta_w)\begin{bmatrix} mrs_t+\varepsilon_W\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_W)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \\ -\varepsilon_w\varphi\tilde{w}_t\left(\dfrac{1}{1-\beta\theta_w}\right) \end{bmatrix}, \\
&= (1-\beta\theta_w)\begin{bmatrix} mrs_t+\varepsilon_w\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_w)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \end{bmatrix}-\varepsilon_w\varphi\tilde{w}_t
\end{aligned}
$$

which can be rewritten as:

$$
(1+\varepsilon_w\varphi)\tilde{w}_t=(1-\beta\theta_w)\begin{bmatrix} mrs_t+\varepsilon_w\varphi w_t+p_t+\beta\theta_w\left(mrs_{t+1}+\varepsilon_w\varphi w_{t+1}+p_{t+1}\right) \\ +(\beta\theta_w)^2\left(mrs_{t+2}+\varepsilon_w\varphi w_{t+2}+p_{t+2}\right)+\cdots \end{bmatrix}.
$$

This expression can be the compact form as follows:

$\tilde{w}_t=\dfrac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\sum_{k=0}^{\infty}(\beta\theta_w)^k \mathrm{E}_t\left(mrs_{t+k}+\varepsilon_w\varphi w_{t+k}+p_{t+k}\right)$. (3-1-33)

Let define

$\mu_t^w \equiv \omega_t - mrs_t$ (3-1-34)

which is the (log) average wage markup in the SOE.

denotes the (log) real wage.

Plugging the definition of the (log) average wage markup into Eq.(3-1-33) yields:

$$\tilde{w}_t = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\sum_{k=0}^{\infty}(\beta\theta_w)^k \mathrm{E}_t\left[(1+\varepsilon_w\varphi)w_{t+k} - \mu_{t+k}^w\right].\ (3\text{-}1\text{-}35)$$

Eq.(3-1-35) can be rewritten as:

$$\begin{aligned}\tilde{w}_t &= \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\mathrm{E}_t\begin{Bmatrix}(1+\varepsilon_w\varphi)w_t - \mu_t^w + \beta\theta_w\left[(1+\varepsilon_w\varphi)w_{t+1} - \mu_{t+1}^w\right] \\ +(\beta\theta_w)^2\left[(1+\varepsilon_w\varphi)w_{t+2} - \mu_{t+2}^w\right] + \cdots\end{Bmatrix} \\ &= \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\left[(1+\varepsilon_w\varphi)w_t - \mu_t^w\right] + \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\mathrm{E}_t\begin{Bmatrix}\beta\theta_w\left[(1+\varepsilon_w\varphi)w_{t+1} - \mu_{t+1}^w\right] \\ +(\beta\theta_w)^2\left[(1+\varepsilon_w\varphi)w_{t+2} - \mu_{t+2}^w\right] \\ +\cdots\end{Bmatrix}\end{aligned}.\ (3\text{-}1\text{-}36)$$

Forwarding Eq.(3-1-36) one period yield:

$$\mathrm{E}_t(\tilde{w}_{t+1}) = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\mathrm{E}_t\begin{Bmatrix}(1+\varepsilon_w\varphi)w_{t+1} - \mu_{t+1}^w + \beta\theta_w\left[(1+\varepsilon_w\varphi)w_{t+2} - \mu_{t+2}^w\right] \\ +(\beta\theta_w)^2\left[(1+\varepsilon_w\varphi)w_{t+3} - \mu_{t+3}^w\right] + \cdots\end{Bmatrix}$$

Multiplying $\beta\theta_w$ both sides of the previous expression yields:

$$\beta\theta_w\mathrm{E}_t(\tilde{w}_{t+1}) = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\mathrm{E}_t\begin{Bmatrix}\beta\theta_w\left[(1+\varepsilon_w\varphi)w_{t+1} - \mu_{t+1}^w\right] + (\beta\theta_w)^2\left[(1+\varepsilon_w\varphi)w_{t+2} - \mu_{t+2}^w\right] \\ +(\beta\theta_w)^3\left[(1+\varepsilon_w\varphi)w_{t+3} - \mu_{t+3}^w\right] + \cdots\end{Bmatrix}$$

Plugging the previous expression into Eq.(3-1-36) yields:

$$\tilde{w}_t = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\left[(1+\varepsilon_w\varphi)w_t - \mu_t^w\right] + \beta\theta_w\mathrm{E}_t(\tilde{w}_{t+1}).\ (3\text{-}1\text{-}37)$$

Total derivative of Eq.(1-1-97) is given by:

$$\begin{aligned}dW_t &= \frac{1}{1-\varepsilon_w}\left(W^{1-\varepsilon_w}\right)^{\frac{1}{1-\varepsilon_w}-1}\left[(1-\varepsilon_w)\theta_w W^{-\varepsilon_w}dW_{t-1} + (1-\varepsilon_w)(1-\theta_w)W^{-\varepsilon_w}d\tilde{W}_t\right] \\ &= \left(W^{1-\varepsilon_w}\right)^{\frac{1-(1-\varepsilon_w)}{1-\varepsilon_w}}W^{-\varepsilon_w}\left[\theta_w dW_{t-1} + (1-\theta_w)d\tilde{W}_t\right] \\ &= \left(W^{1-\varepsilon_w}\right)^{\frac{\varepsilon_w}{1-\varepsilon_w}}W^{-\varepsilon_w}\left[\theta_w dW_{t-1} + (1-\theta_w)d\tilde{W}_t\right] \\ &= \theta_w dW_{t-1} + (1-\theta_w)d\tilde{W}_t\end{aligned}.$$

By dividing both sides of the previous expression W_H, we have:

$$\frac{dW_t}{W} = \theta_w\frac{dW_{t-1}}{W} + (1-\theta_w)\frac{d\tilde{W}_t}{W},$$

which can be rewritten as:

$$w_t = \theta_w w_{t-1} + (1-\theta_w)\tilde{w}_t.\ (3\text{-}1\text{-}38)$$

Eq.(3-1-38) can be rewritten as:

$$\tilde{w}_t = \frac{1}{1-\theta_w} w_t - \frac{\theta_w}{1-\theta_w} w_{t-1}.$$

Plugging the previous expression into Eq. (3-1-37) yields:

$$\frac{1}{1-\theta_w} w_t - \frac{\theta_w}{1-\theta_w} w_{t-1} = \frac{1-\beta\theta_w}{1+\varepsilon_w\varphi}\left[(1+\varepsilon_w\varphi)w_t - \mu_t^w\right]$$
$$+\beta\theta_w \mathrm{E}_t\left(\frac{1}{1-\theta_w} w_{t+1} - \frac{\theta_w}{1-\theta_w} w_t\right),$$

Which can be rewritten as:

$$\begin{aligned}
w_t - \theta_w w_{t-1} &= (1-\beta\theta_w)(1-\theta_w)w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w \\
&\quad +\beta\theta_w \mathrm{E}_t\left(w_{t+1} - \theta_w w_t\right) \\
&= (1-\beta\theta_w)w_t - \theta_w(1-\beta\theta_w)w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w \\
&\quad +\beta\theta_w \mathrm{E}_t\left(w_{t+1}\right) - \beta\theta_w^2 w_t \\
&= (1-\beta\theta_w)w_t - \theta_w w_t + \beta\theta_w^2 w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w \\
&\quad +\beta\theta_w \mathrm{E}_t\left(w_{t+1}\right) - \beta\theta_w^2 w_t \\
&= w_t - \theta_w w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w \\
&\quad +\beta\theta_w \mathrm{E}_t\left(w_{t+1}\right) - \beta\theta_w w_t
\end{aligned}$$

Subtracting $w_{H,t}$ from both sides of the previous expression yields:

$$-\theta_w w_{t-1} = -\theta_w w_t - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w + \beta\theta_w \mathrm{E}_t\left(w_{t+1} - w_t\right),$$

which can be rewritten as:

$$\theta_w\left(w_t - w_{t-1}\right) = \beta\theta_w \mathrm{E}_t\left(w_{t+1} - w_t\right) - \frac{(1-\beta\theta_w)(1-\theta_w)}{1+\varepsilon_w\varphi}\mu_t^w.$$

By dividing both sides by θ_w, we have:

$\pi_t^w = \beta \mathrm{E}_t\left(\pi_{t+1}^w\right) - \kappa_w \mu_t^w$, (3-1-39)

with $\kappa_w \equiv \dfrac{(1-\beta\theta_w)(1-\theta_w)}{\theta_w(1+\varepsilon_w\varphi)}$ where:

$\pi_t^w \equiv w_t - w_{t-1}$, (3-1-40)

denotes the wage inflation.

There is a relationship between the changes in the real wage and the gap between the wage inflation and the domestic inflation as follows:

$$\begin{aligned} w_t^r - w_{t-1}^r &= w_t - p_{H,t} - \left(w_{t-1} - p_{H,t-1}\right) \\ &= w_t - w_{t-1} - \left(p_{H,t} - p_{H,t-1}\right). \\ &= \pi_t^W - \pi_{H,t} \end{aligned}$$

3.1.9 UIP

The log-linearized UIP $R_t = \mathrm{E}_t\left(E_{t+1}/E_t\right)R_t^*$ is given by

$\hat{r}_t = \hat{r}_t^* + \mathrm{E}_t\left(e_{t+1}\right) - e_t$. (3-1-43).

Plugging Eq.(3-1-43) into Eq.(3-1-42) yields:

3.1.10 Fiscal Policy Rules

Fiscal feed back rule in Mahdavi (2004) can be rewritten as:

$$\frac{SP_t}{SP} = \phi_B \frac{B_{t-1}}{B} \frac{\beta}{1-\beta},$$

where we use $SP = \left(\dfrac{1-\beta}{\beta}\right)B$ being Eq.(2-24). The previous expression can be rewritten as:

$$\frac{SP_t}{SP} - 1 = \frac{\beta\phi_B}{1-\beta}\frac{B_{t-1}}{B} - 1.$$

Then:

$$
\begin{aligned}
\ln\left(\frac{SP_t}{SP}\right) &= \phi_b \frac{B_{t-1}}{B} - 1 \\
&= \phi_b \left(\frac{B_{t-1}}{B} - \frac{1}{\phi_b} + 1 - 1\right) \\
&= \phi_b \left(\frac{B_{t-1}}{B} - 1 - \frac{1}{\phi_b} + 1\right) \\
&= \phi_b \left(\frac{B_{t-1}}{B} - 1\right) - \phi_b \left(\frac{1}{\phi_b} - 1\right) \\
&= \phi_b \ln\left(\frac{B_{t-1}}{B}\right) + (\phi_b - 1)
\end{aligned} ,
$$

which can be rewritten as:

$sp_t = \phi_b b_{H,t-1} + (\phi_b - 1)$. (3-1-44) [(47) in the text]

Plugging Eq.(3-1-22) into (3-1-44) yields:

$\hat{\tau}_t = \frac{(1-\beta)\sigma_B \phi_b}{\beta} b_{H,t-1} + \frac{\upsilon(1-\beta)\sigma_B}{\beta} s_t - \tau y_t + \sigma_G g_t + \hat{\zeta}_t + \frac{(1-\beta)\sigma_B(\phi_b - 1)}{\beta}$. [(48) in the text]

We introduce tax shock ψ_t as follows:

$\tau_t = \frac{(1-\beta)\sigma_B \phi_b}{\beta} b_{H,t-1} + \frac{\upsilon(1-\beta)\sigma_B}{\beta} s_t - \tau y_t + \sigma_G g_t + \hat{\zeta}_t + \frac{(1-\beta)\sigma_B(\phi_b - 1)}{\beta} + \psi_t$. (3-1-45) [(49) in the text]

4 Welfare Cost Function and Welfare Relevant Output Gap

4.1 Deriving the Second-order Approximated Utility Function with Linear Terms

The period utility function is given by Eq.(1-1-85), namely:

$\tilde{U}_{t+k} \equiv \ln C_{t+k} - \frac{1}{1+\varphi} \int_0^1 N_{t+k|t}(j)^{1+\varphi} dj$. (1-1-85)

In equilibrium,

$$
\begin{aligned}
N_t(j) &= N_t(j) \\
&= \left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W} N_t ,
\end{aligned}
$$

Is applied. Plugging this expression into Eq.(1-1-85) yields:

$$\tilde{U}_{t+k} \equiv \ln C_{t+k} - \frac{1}{1+\varphi} N_{t+k}^{1+\varphi} \int_0^1 \left(\frac{W_{t+k}(j)}{W_{t+k}} \right)^{-\varepsilon_w(1+\varphi)} dj .$$

Let define $\left(\Delta_t^w\right)^{1+\varphi} \equiv \int_0^1 \left(\frac{W_{H,t}(j)}{W_{H,t}} \right)^{-\varepsilon_w(1+\varphi)} dj$. Plugging this definition into the previous expression, we have:

$$\tilde{U}_{H,t+k} \equiv \ln C_{t+k} - \frac{1}{1+\varphi} N_{t+k}^{1+\varphi} \left(\Delta_{t+k}^w\right)^{1+\varphi} . \text{ (4-1-1)}$$

Second-order expansion (percentage deviation from steady state in terms of marginal utility of consumption) of Eq.(4-1) is given by:

$$\begin{aligned}\frac{\tilde{U}_{t+k} - \tilde{U}}{\tilde{U}_C C} &= c_{t+k} + \frac{1}{2} c_{t+k}^2 + \frac{1}{2} \frac{\tilde{U}_{CC}}{\tilde{U}_C} C c_{t+k}^2 + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} \left(n_{t+k} + \frac{1}{2} n_{t+k}^2 \right) + \frac{1}{2} \frac{\tilde{U}_{NN}}{\tilde{U}_C} \frac{N^2}{C} n_{t+k}^2 \\ &\quad + \frac{\tilde{U}_Z}{\tilde{U}_C} \frac{1}{C} \ln \Delta_{t+k}^w + o\left(\|\xi\|^3\right)\end{aligned} .$$

Note that $\ln \Delta_{t+k}^W$ is $o\left(\|\xi\|^2\right)$.

Plugging $\tilde{U}_C = C^{-1}$, $\tilde{U}_{CC} = -C^{-2}$, $\tilde{U}_N = -N^{\varphi}$, $\tilde{U}_{NN} = -\varphi N^{\varphi-1}$ and $\tilde{U}_Z = -N^{1+\varphi}$ into the previous expression yields:

$$\begin{aligned}\frac{U_t - U}{U_C C} &= c_t + \frac{1}{2} c_t^2 + \frac{1}{2} \frac{\left(-C^{-2}\right)}{C^{-1}} C c_t^2 + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} \left(n_{t+k} + \frac{1}{2} n_{t+k}^2 \right) + \frac{1}{2} \frac{\left(-\varphi N^{\varphi-1}\right)}{\tilde{U}_C} \frac{N^2}{C} n_{t+k}^2 \\ &\quad - \frac{N^{1+\varphi}}{\tilde{U}_C} \frac{1}{C} z_{H,t+k}^W + o\left(\|\xi\|^3\right) \\ &= c_t + \frac{1}{2} c_t^2 - \frac{1}{2} \frac{C^2}{C^2} c_t^2 + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} \left(n_{t+k} + \frac{1}{2} n_{t+k}^2 \right) + \varphi \frac{1}{2} \frac{\left(-N^{\varphi}\right)}{\tilde{U}_C} \frac{1}{N} \frac{N^2}{C} n_{t+k}^2 \\ &\quad + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} z_{H,t+k}^W + o\left(\|\xi\|^3\right) \\ &= c_t + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} n_{t+k} + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} \frac{1}{2} n_{t+k}^2 + \varphi \frac{1}{2} \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} n_{t+k}^2 + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} \ln \Delta_{t+k}^W + o\left(\|\xi\|^3\right) \\ &= c_t + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} n_{t+k} + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} \frac{1}{2} (1+\varphi) n_{t+k}^2 + \frac{\tilde{U}_N}{\tilde{U}_C} \frac{N}{C} \ln \Delta_{t+k}^W + o\left(\|\xi\|^3\right)\end{aligned} .$$

Plugging $-\frac{\tilde{U}_N}{\tilde{U}_C} \equiv 1 - \Phi$ into the previous expression yields:

$$
\begin{aligned}
\frac{\tilde{U}_t - \tilde{U}}{\tilde{U}_C C} &= c_t - (1-\Phi)\frac{N}{C}n_{t+k} - (1-\Phi)\frac{N}{C}\frac{1+\varphi}{2}n_{t+k}^2 - (1-\Phi)\frac{N}{C}\ln\Delta_{t+k}^w + o\left(\|\xi\|^3\right) \\
&= c_t + \Phi\frac{N}{C}n_{t+k} - \frac{N}{C}n_{t+k} - (1-\Phi)\frac{N}{C}\frac{1+\varphi}{2}n_{t+k}^2 - (1-\Phi)\frac{N}{C}\ln\Delta_{t+k}^w + o\left(\|\xi\|^3\right). \\
&= c_t + \Phi\frac{N}{C}n_{t+k} - \frac{N}{C}\left[n_{t+k} + \frac{(1-\Phi)(1+\varphi)}{2}n_{t+k}^2 + (1-\Phi)\ln\Delta_{t+k}^w\right] + o\left(\|\xi\|^3\right)
\end{aligned}
$$

Plugging $n_{t+k} = y_{H,t+k} - a_{t+k} + \ln\Delta_{t+k}^p$ into the last line in the previous expression yields:

$$
\begin{aligned}
\frac{\tilde{U}_t - \tilde{U}}{\tilde{U}_C C} &= c_{t+k} + \frac{\Phi}{\sigma_C}\left(y_{t+k} + \ln\Delta_{t+k}^p\right) \\
&\quad - \frac{1}{\sigma_C}\begin{bmatrix}\left(y_{t+k} + \ln\Delta_{t+k}^p\right) \\ + \frac{(1-\Phi)(1+\varphi)}{2}n_{t+k}^2 + (1-\Phi)\ln\Delta_{t+k}^w\end{bmatrix} + o\left(\|\xi\|^3\right) \qquad , (4\text{-}5\text{-}3) \\
&= c_{t+k} - \frac{1-\Phi}{\sigma_C}y_{t+k} - \frac{1}{\sigma_C}\begin{bmatrix}\frac{(1-\Phi)(1+\varphi)}{2}n_{t+k}^2 \\ + (1-\Phi)\ln\Delta_{t+k}^p + (1-\Phi)\ln\Delta_{t+k}^w\end{bmatrix} + o\left(\|\xi\|^3\right)
\end{aligned}
$$

where we use $N/C = (C/Y)^{-1} = \sigma_C^{-1}$ and $\ln\Delta_{t+k}^p = \ln\left[\int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} dh\right]$.

4.2 Second-order Approximation of Price and Wage Dispersions

Relative price of good *h* can be approximated as follows:

$$
\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon} = 1 + (1-\varepsilon_p)\hat{p}_{H,t}(i) + \frac{1}{2}(1-\varepsilon_p)^2\hat{p}_{H,t}(i)^2 + o\left(\|\xi\|^3\right), \ (4\text{-}5\text{-}12)
$$

with $\hat{p}_{H,t}(i) \equiv \ln P_{H,t}(i) - \ln P_{H,t}$.

The price index $P_{H,t} \equiv \left[\int_0^1 P_{H,t}(i)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}$ implies:

$$
1 = \frac{1}{P_{H,t}}\left[\int_0^1 P_{H,t}(i)^{1-\varepsilon_p} di\right]^{\frac{1}{1-\varepsilon_p}} .
$$

Hence, we have:

$$
\begin{aligned}
1^{1-\varepsilon} &= \left(\frac{1}{P_{H,t}}\right)^{1-\varepsilon_p} \int_0^{\tilde{\alpha}} P_t(i)^{1-\varepsilon_p} di \\
&= \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon_p} di \qquad , \\
&= 1
\end{aligned}
$$

which implies that:

$$
\begin{aligned}
E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon_p} &= \int_0^1 \left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon_p} di \\
&= 1
\end{aligned} \quad . \text{ (4-5-13)}
$$

Eq.(4-5-12) can be rewritten as:

$$
\left(1-\varepsilon_p\right)\hat{p}_{H,t}(i) = -\frac{1}{2}\left(1-\varepsilon_p\right)^2 \hat{p}_{H,t}(i)^2 - \left[1-\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon_p}\right] + o\left(\|\xi\|^3\right),
$$

Taking conditional expectation, The previous expression can be rewritten as:

$$
\begin{aligned}
E_i\left[\hat{p}_{H,t}(i)\right] &= -\frac{1-\varepsilon_p}{2} E_i\left[\hat{p}_{H,t}(i)^2\right] - \frac{1}{1-\varepsilon_p}\left[1-E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{1-\varepsilon}\right] + o\left(\|\xi\|^3\right) \\
&= -\frac{1-\varepsilon_p}{2} E_i\left[\hat{p}_{H,t}(i)^2\right] + o\left(\|\xi\|^3\right)
\end{aligned} \quad , \text{ (4-5-14)}
$$

where we use Eq.(4-5-13) to derive the second line.

Second-order approximation to $\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}$ yields:

$$
\begin{aligned}
\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} &= \exp\left[-\varepsilon_p \hat{p}_{H,t}(i)\right] \\
&= 1-\varepsilon_p \hat{p}_{H,t}(i) + \frac{\varepsilon^2}{2} \hat{p}_{H,t}(i)^2 + o\left(\|\xi\|^3\right)
\end{aligned} \quad .
$$

Taking conditional expectation on the previous expression yields:

$$
E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon_p} = 1-\varepsilon_p \mathrm{E}_i\left[\hat{p}_{H,t}(i)\right] + \frac{\varepsilon_p^2}{2} \mathrm{E}_i\left[\hat{p}_{H,t}(i)^2\right] + o\left(\|\xi\|^3\right). \text{ (4-5-15)}
$$

Plugging Eq.(4-5-14) into (4-5-15) yields:

$$E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon} = 1-\varepsilon_p\left\{-\frac{1-\varepsilon}{2}E_h\left[\hat{p}_{H,t}(i)^2\right]\right\}+\frac{\varepsilon_p^2}{2}\mathrm{E}_i\left[\hat{p}_{H,t}(i)^2\right]+o\left(\|\xi\|^3\right)$$
$$=1+\frac{\varepsilon_p\left[\left(1-\varepsilon_p\right)+\varepsilon_p\right]}{2}\mathrm{E}_i\left[\hat{p}_{H,t}(i)^2\right]+o\left(\|\xi\|^3\right)$$
$$=1+\frac{\varepsilon_p}{2}\mathrm{E}_i\left[\hat{p}_{H,t}(i)^2\right]+o\left(\|\xi\|^3\right)$$
$$=1+\frac{\varepsilon_p}{2}\mathrm{var}_i\left[\hat{p}_{H,t}(i)\right]+o\left(\|\xi\|^3\right) . \quad (4\text{-}14\text{-}16)$$

Notice that $E_i\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}=\int_0^1\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}di$ and $\Delta_t^p=\int_0^1\left(\frac{P_{H,t}(i)}{P_{H,t}}\right)^{-\varepsilon}di$. By using these facts, Eq.(4-14-16) can be rewritten as:

$$\ln\Delta_t^p=\ln\left\{1+\frac{\varepsilon_p}{2}\mathrm{var}_i\left[\hat{p}_{H,t}(i)\right]\right\}+o\left(\|\xi\|^3\right)$$
$$\approx\frac{\varepsilon_p}{2}\mathrm{var}_i\left[\hat{p}_{H,t}(i)\right]+o\left(\|\xi\|^3\right) . \quad (4\text{-}2\text{-}6)$$

Further, as shown in Woodford (2003), $z_{H,t}$ can be rewritten as:

$$\sum_{k=0}^{\infty}\beta^{\mathrm{t}}\mathrm{E}_t\left(\ln\Delta_t^p\right)=\frac{\varepsilon_p}{2\kappa_p}\sum_{k=0}^{\infty}\beta^{\mathrm{t}}\mathrm{E}_t\left(\pi_{H,t}^2\right), \quad (4\text{-}6\text{-}2)$$

as long as Eq.(4-5-10) is applicable.

Relative wage *j* can be approximated as follows:

$$\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W}=1+\left(1-\varepsilon_W\right)\hat{w}_t(j)+\frac{1}{2}\left(1-\varepsilon_W\right)^2\hat{w}_t(j)^2+o\left(\|\xi\|^3\right), \quad (4\text{-}6\text{-}3)$$

with $\hat{w}_t(j)\equiv\ln W_t(j)-\ln W_t$.

The wage index $W_t\equiv\left[\int_0^1 W_t(j)^{1-\varepsilon_W}dj\right]^{\frac{1}{1-\varepsilon_W}}$ implies:

$$1=\frac{1}{W_t}\left[\int_0^1 W_t(j)^{1-\varepsilon_W}dj\right]^{\frac{1}{1-\varepsilon_W}}.$$

Hence, we have:

$$
\begin{aligned}
1^{1-\varepsilon_W} &= \left(\frac{1}{W_t}\right)^{1-\varepsilon_W} \int_0^1 W_t(j)^{1-\varepsilon_W}\, dj \\
&= \int_0^1 \left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W} dj \qquad , \\
&= 1
\end{aligned}
$$

which implies that:

$$
\begin{aligned}
E_j\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W} &= \int_0^1 \left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W} dj \\
&= 1
\end{aligned}
. \text{ (4-14-17)}
$$

Eq.(4-6-3) can be rewritten as:

$$
(1-\varepsilon_W)\hat{w}_t(j) = -\frac{1}{2}(1-\varepsilon_W)^2 \hat{w}_t(j)^2 - \left[1-\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W}\right] + o\left(\|\xi\|^3\right),
$$

Taking conditional expectation, The previous expression can be rewritten as:

$$
\begin{aligned}
\mathrm{E}_j\left[\hat{w}_t(j)\right] &= -\frac{1-\varepsilon_W}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right] - \frac{1}{1-\varepsilon_W}\left[1-\mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{1-\varepsilon_W}\right] + o\left(\|\xi\|^3\right) \\
&= -\frac{1-\varepsilon_W}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right] + o\left(\|\xi\|^3\right)
\end{aligned}
, \text{ (4-2-10)}
$$

where we use Eq.(4-14-17) to derive the second line.

Second-order approximation to $\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W}$ yields:

$$
\begin{aligned}
\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} &= \exp\left[-\varepsilon_W(1+\varphi)\hat{w}_t(j)\right] \\
&= 1-\varepsilon_W(1+\varphi)\hat{w}_t(j) + \frac{\varepsilon_W^2(1+\varphi)^2}{2}\hat{w}_t(j)^2 + o\left(\|\xi\|^3\right)
\end{aligned}
.
$$

Taking conditional expectation on the previous expression yields:

$$
\mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} = 1-\varepsilon_W(1+\varphi)\mathrm{E}_j\left[\hat{w}_t(j)\right] + \frac{\varepsilon_W^2(1+\varphi)^2}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right] + o\left(\|\xi\|^3\right). \text{ (4-2-11)}
$$

Plugging Eq.(4-2-11) into (4-2-10) yields:

$$\mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} = 1-\varepsilon_W(1+\varphi)\left\{-\frac{1-\varepsilon_W}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]\right\}+\frac{\varepsilon_W^2(1+\varphi)^2}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]$$
$$+o\left(\|\xi\|^3\right)$$
$$=1+\frac{\varepsilon_W(1+\varphi)\left[(1-\varepsilon_W)+\varepsilon_W(1+\varphi)\right]}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]+o\left(\|\xi\|^3\right) \quad . \text{(4-2-12)}$$
$$=1+\frac{\varepsilon_W(1+\varphi)(1+\varepsilon_W\varphi)}{2}\mathrm{E}_j\left[\hat{w}_t(j)^2\right]+o\left(\|\xi\|^3\right)$$
$$=1+\frac{\varepsilon_W(1+\varphi)(1+\varepsilon_W\varphi)}{2}\mathrm{var}_j\left[\hat{w}_t(j)\right]+o\left(\|\xi\|^3\right)$$

Notice that $\mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} = \int_0^1\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} dj$ and $\left(\Delta_t^w\right)^{1+\varphi} \equiv \int_0^1\left(\frac{W_{H,t}(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} dj$. By using these facts, Eq.(4-2-12) can be rewritten as:

$$\ln\left(\Delta_t^w\right)^{1+\varphi} = \ln\int_0^1\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)} dj$$
$$=\ln\mathrm{E}_j\left(\frac{W_t(j)}{W_t}\right)^{-\varepsilon_W(1+\varphi)}$$
$$=\ln\left\{1+\frac{\varepsilon_W(1+\varphi)(1+\varepsilon_W\varphi)}{2}\mathrm{var}_j\left[\hat{w}_t(j)\right]\right\}+o\left(\|\xi\|^3\right)$$
$$\approx\frac{\varepsilon_W(1+\varphi)(1+\varepsilon_W\varphi)}{2}\mathrm{var}_j\left[\hat{w}_t(j)\right]+o\left(\|\xi\|^3\right) .$$

The previous expression can be rewritten as:

$$\ln\left(\Delta_t^w\right)^{1+\varphi} = \frac{\varepsilon_W(1+\varepsilon_W\varphi)}{2}\mathrm{var}_j\left[\hat{w}_{H,t}(j)\right]+o\left(\|\xi\|^3\right). \text{ (4-2-12)}$$

Further, as shown in Woodford (2003), $\ln\Delta_{t+k}^w$ can be rewritten as:

$$\sum_{k=0}^{\infty}\beta^{\mathrm{t}}\mathrm{E}_t\left(\ln\Delta_{t+k}^w\right) = \frac{\varepsilon_W(1+\varepsilon_W\varphi)}{2\kappa_W}\sum_{k=0}^{\infty}\beta^{\mathrm{t}}\mathrm{E}_t\left(\pi_{H,t}^W\right)^2 . \text{ (4-2-13)}$$

Iterating Eq.(4-5-3) yields:

$$\tilde{\mathcal{W}} = \sum_{k=0}^{\infty} \beta^t \mathrm{E}_t \left\{ c_{t+k} - \frac{1-\Phi}{\sigma_C} y_{t+k} - \frac{1}{\sigma_C} \left[\begin{array}{l} \frac{(1-\Phi)(1+\varphi)}{2} n_{t+k}^2 \\ +(1-\Phi)\ln\Delta_{t+k}^p + (1-\Phi)\ln\Delta_{t+k}^w \end{array} \right] \right\}.$$

$$+o\left(\|\xi\|^3\right)$$

Plugging Eqs. (4-6-2) and (4-2-13) yields:

$$\tilde{\mathcal{W}} = \sum_{k=0}^{\infty} \beta^t \mathrm{E}_t \left[\begin{array}{l} c_{t+k} - \frac{1-\Phi}{\sigma_C} y_{t+k} - \frac{(1-\Phi)(1+\varphi)}{2\sigma_C} n_{H,t+k}^2 \\ -\frac{\varepsilon_p(1-\Phi)}{2\kappa_p\sigma_C} \pi_{H,t+k}^2 - \frac{\varepsilon_w(1+\varepsilon_w\varphi)(1-\Phi)}{2\kappa_w\sigma_C} \left(\pi_{t+k}^W\right)^2 \end{array} \right] + o\left(\|\xi\|^3\right), \text{ (4-2-14)}$$

with $\tilde{\mathcal{W}}_H \equiv \sum_{k=0}^{\infty} \beta^k \mathrm{E}_t \left(\frac{\tilde{U}_{t+k} - \tilde{U}}{\tilde{U}_C C} \right)$.

Eq.(3-1-8)’ can be rewritten as:

$$c_t = \frac{1}{(1-\upsilon)\sigma_C} y_t - \frac{\eta\upsilon(2-\upsilon)}{1-\upsilon} s_t - \frac{\upsilon}{1-\upsilon} z_{1,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C} g_t \quad \text{(4-2-15)}$$

Plugging Eqs.(4-2-15) into Eq.(4-2-14) yields:

$$\tilde{\mathcal{W}} = \sum_{k=0}^{\infty} \beta^t \mathrm{E}_t \left\{ \begin{array}{l} \frac{\Phi}{\sigma_C} y_{t+k} - \frac{\eta\upsilon(2-\upsilon)}{1-\upsilon} s_{t+k} - \frac{(1-\Phi)(1+\varphi)}{2\sigma_C} n_{H,t+k}^2 \\ -\frac{\varepsilon_p(1-\Phi)}{2\kappa_p\sigma_C} \pi_{H,t+k}^2 - \frac{\varepsilon_w(1+\varepsilon_w\varphi)(1-\Phi)}{2\kappa_w\sigma_C} \left(\pi_{t+k}^W\right)^2 \end{array} \right\} + o\left(\|\xi\|^3\right), \text{ (4-2-17)}$$

which is applicable for the SOE withOUT default risk.

4.3 Second-order Approximation of the AS Equation

Second-order approximated AS equation is given by:

$$\bar{\nu} = \kappa_p \sum_{k=0}^{\infty} \beta^k \mathrm{E}_t \left\{ \begin{array}{l} w_{t+k}^r - \frac{2\tau}{1-\tau} \hat{\tau}_{t+k} + \frac{\tau}{1-\tau} c_{t+k} - \frac{\tau}{1-\tau} y_{t+k} - \frac{\tau}{1-\tau} x_{H,t+k} + \frac{1}{2}\left(w_{t+k}^r\right)^2 \\ + \frac{\tau}{2(1-\tau)} c_{t+k}^2 - \frac{\tau}{2(1-\tau)} y_{t+k}^2 - \frac{\tau}{2(1-\tau)} x_{H,t+k}^2 - w_{t+k}^r a_{t+k} - c_{t+k} w_{t+k}^r \\ + c_{t+k} a_{t+k} + y_{t+k} w_{t+k}^r - y_{t+k} a_{H,t+k} + x_{t+k} w_{H,t+k}^r - x_{H,t+k} a_{t+k} \end{array} \right\}, \text{ (4-}$$

$$+\frac{\kappa_p \varepsilon_p}{2} \sum_{k=0}^{\infty} \beta^k \pi_{H,t+k}^2 + \text{s.o.t.i.p.} + o\left(\|\xi\|^3\right)$$

3-1)

with $\omega_z \equiv \sigma_c(\eta-1)(1-\alpha)-1$.

4.4 Second-order Approximation of the Wage Equation

Eq.(1-1-95) can be rewritten as:

$$\frac{\tilde{W}_{H,t}}{W_{H,t}} = \frac{M^W \sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left(MRS_{H,t+k|t} N_{H,t+k|t} C_{t+k}^{-1} \right)}{\sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left[N_{H,t+k|t} \left(P_{t+k} C_{t+k} \right)^{-1} \right] W_{H,t}} . \text{ (4-4-1)}$$

Let define:

$$K_{H,t}^W \equiv M^W \sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left(MRS_{H,t+k|t} N_{H,t+k|t} C_{t+k}^{-1} \right), \text{ (4-4-2)}$$

$$F_{H,t}^W \equiv \sum_{k=0}^{\infty} (\beta\theta_W)^k \mathrm{E}_t \left[N_{H,t+k|t} \left(P_{t+k} C_{t+k} \right)^{-1} \right] W_{H,t} . \text{ (4-4-3)}$$

By plugging Eqs.(4-4-2) and (4-4-3) into Eq.(4-6-3) yields:

$$\frac{\tilde{W}_{H,t}}{W_{H,t}} = \frac{K_{H,t}^W}{F_{H,t}^W} . \text{ (4-4-4)}$$

Multiplying Eq.(4-4-4) by $W_{H,t}$ yields:

$$\tilde{W}_{H,t} = \frac{W_{H,t} K_{H,t}^W}{F_{H,t}^W}$$

Plugging the previous expression Eq.(1-1-97) yields:

$$W_{H,t} = \left[\theta_W W_{H,t-1}^{1-\varepsilon_W} + (1-\theta_W) \left(\frac{W_{H,t} K_{H,t}^W}{F_{H,t}^W} \right)^{1-\varepsilon_W} \right]^{\frac{1}{1-\varepsilon_W}}, \text{ which can be rewritten as:}$$

$$\begin{aligned} W_{H,t}^{1-\varepsilon_W} &= (1-\theta_W) \left(\frac{W_{H,t} K_{H,t}^W}{F_{H,t}^W} \right)^{1-\varepsilon_W} + \theta_W W_{H,t-1}^{1-\varepsilon_W} \\ &= (1-\theta_W) W_{H,t}^{1-\varepsilon_W} \left(K_{H,t}^W \right)^{1-\varepsilon_W} \left(F_{H,t}^W \right)^{-(1-\varepsilon_W)} + \theta_W W_{H,t-1}^{1-\varepsilon_W} \end{aligned} .$$

Dividing the previous expression by $W_{H,t}^{1-\varepsilon_W}$ yields:

$$\begin{aligned} 1 &= (1-\theta_W) \left(K_{H,t}^W \right)^{1-\varepsilon_W} \left(F_{H,t}^W \right)^{-(1-\varepsilon_W)} + \theta_W \left(\frac{W_{H,t-1}}{W_{H,t}} \right)^{1-\varepsilon_W}, \\ &= (1-\theta_W) \left(K_{H,t}^W \right)^{1-\varepsilon_W} \left(F_{H,t}^W \right)^{-(1-\varepsilon_W)} + \theta_W \left(\Pi_{H,t}^W \right)^{-(1-\varepsilon_W)} \end{aligned}$$

which can be rewritten as:

$$\frac{1}{1-\theta_W}\left[1-\theta_W\left(\Pi_{H,t}^W\right)^{-(1-\varepsilon_W)}\right]=\left(\frac{F_{H,t}^W}{K_{H,t}^W}\right)^{\varepsilon_W-1}. \text{ (4-4-6)}$$

Taking logarithm both sides of Eq.(4-4-6) yields:

$$\ln\left[\frac{1}{1-\theta_W}-\frac{\theta_W}{1-\theta_W}\left(\Pi_{H,t}^W\right)^{-(1-\varepsilon_W)}\right]=\left(\varepsilon_W-1\right)\left(\ln F_{H,t}^W-\ln K_{H,t}^W\right),$$

which can be rewritten as:

$$-\ln\left[\frac{1}{1-\theta_W}-\frac{\theta_W}{1-\theta_W}\left(\Pi_{H,t}^W\right)^{-(1-\varepsilon_W)}\right]=\left(\varepsilon_W-1\right)\left(\ln K_{H,t}^W-\ln F_{H,t}^W\right). \text{ (4-4-7)}$$

Second-order approximation of the LHS of Eq.(4-4-6) is given by:

$$-\ln\left[\frac{1}{1-\theta_W}-\frac{\theta_W}{1-\theta_W}\left(\Pi_{H,t}^W\right)^{-(1-\varepsilon_W)}\right]=-\frac{\theta_W\left(\varepsilon_W-1\right)}{1-\theta_W}\left[\pi_{H,t}^W+\frac{\varepsilon_W-1}{2\left(1-\theta_W\right)}\left(\pi_{H,t}^W\right)^2\right]+o\left(\left\|\xi\right\|^3\right).$$

Eq.(4-4-3) can be rewritten as:

$$\begin{aligned}
F_{H,t}^W&=W_{H,t}\left[N_{H,t|t}\left(P_tC_t\right)^{-1}+\beta\theta_W N_{H,t+1|t}\left(P_{t+1}C_{t+1}\right)^{-1}+\left(\beta\theta_W\right)^2N_{H,t+2|t}\left(P_{t+2}C_{t+2}\right)^{-1}+\cdots\right]\\
&=W_{H,t}\left[\begin{array}{l}\left(\frac{\tilde W_t}{W_{H,t}}\right)^{-\varepsilon_W}N_{H,t+k}\left(P_tC_t\right)^{-1}+\beta\theta_W\left(\frac{\tilde W_t}{W_{H,t+1}}\right)^{-\varepsilon_W}N_{H,t+k}\left(P_{t+1}C_{t+1}\right)^{-1}\\+\left(\beta\theta_W\right)^2\left(\frac{\tilde W_t}{W_{H,t+2}}\right)^{-\varepsilon_W}N_{H,t+k}\left(P_{t+2}C_{t+2}\right)^{-1}+\cdots\end{array}\right]\\
&=\left[\begin{array}{l}C_t^{-1}\left(\frac{\tilde W_{H,t}}{W_{H,t}}\right)^{-\varepsilon_W}N_{H,t}\frac{P_{H,t}}{P_t}\frac{W_{H,t}}{P_{H,t}}+\beta\theta_W C_{t+1}^{-2}\left(\frac{\tilde W_{H,t}}{W_{H,t}}\frac{W_{H,t}}{W_{H,t+1}}\right)^{-\varepsilon_W}N_{H,t+k}\frac{W_{H,t}}{W_{H,t+1}}\frac{W_{H,t+1}}{P_{H,t+1}}\frac{P_{H,t+1}}{P_{t+1}}\\+\left(\beta\theta_W\right)^2C_{t+2}^{-1}\left(\frac{\tilde W_{H,t}}{W_{H,t}}\frac{W_{H,t}}{W_{H,t+1}}\frac{W_{H,t+1}}{W_{H,t+2}}\right)^{-\varepsilon_W}N_{H,t+k}\frac{W_{H,t}}{W_{H,t+1}}\frac{W_{H,t+1}}{W_{H,t+2}}\frac{W_{H,t+2}}{P_{H,t+2}}\frac{P_{H,t+2}}{P_{t+2}}+\cdots\end{array}\right],\\
&=\left[\begin{array}{l}C_t^{-1}\left(\tilde X_{H,t}^W\right)^{-\varepsilon_W}N_{H,t}X_{H,t}W_{H,t}^r+\beta\theta_W C_{t+1}^{-1}\left(\tilde X_{H,t}^W\right)^{-\varepsilon_W}\left(\Pi_{H,t+1}^W\right)^{\varepsilon_W-1}N_{H,t+k}X_{H,t+1}W_{H,t+1}^r\\+\left(\beta\theta_W\right)^2C_{t+1}^{-1}\left(\tilde X_{H,t}^W\right)^{-\varepsilon_W}\left(\Pi_{H,t+1}^W\Pi_{H,t+2}^W\right)^{\varepsilon_W-1}N_{H,t+2}X_{H,t+2}W_{H,t+2}^r+\cdots\end{array}\right]
\end{aligned}$$

with $\tilde X_{H,t}^W\equiv\frac{\tilde W_t}{W_{H,t}}$, $W_{H,t}^r\equiv\frac{W_{H,t}}{P_{H,t}}$ and $\Pi_{H,t}^W\equiv\frac{W_{H,t}}{W_{H,t-1}}$ where we use Eq.(1-1-89). The last line can be rewritten as compact form as:

$$F_{H,t}^{w}=\sum_{k=0}^{\infty}\left(\beta\theta_{w}\right)^{k}\mathrm{E}_{t}\left[\left(\tilde{X}_{H,t}^{w}\right)^{-\varepsilon_{w}}C_{t+k}^{-1}N_{H,t+k}X_{H,t+k}W_{H,t+k}^{r}\left(\prod_{s=1}^{k}\Pi_{H,t+s}^{w}\right)\right]^{\varepsilon_{w}-1}.$$

Eq.(4-4-2) can be rewritten as:

$$\begin{aligned}K_{H,t}^{w}&=M^{w}\left[MRS_{H,t|t}N_{H,t|t}C_{t}^{-1}+\beta\theta_{w}MRS_{H,t+1|t}N_{H,t+1|t}C_{t+1}^{-1}+\left(\beta\theta_{w}\right)^{k}MRS_{H,t+2|t}N_{H,t+2|t}C_{t+2}^{-1}+\cdots\right]\\
&=M^{w}\left[N_{H,t|t}^{1+\varphi}+\beta\theta_{w}N_{H,t+1|t}^{1+\varphi}+\left(\beta\theta_{w}\right)^{k}N_{H,t+2|t}^{1+\varphi}+\cdots\right]\\
&=M^{w}\left[\left(\frac{\tilde{W}_{H,t}}{W_{H,t}}\right)^{-\varepsilon_{w}(1+\varphi)}N_{H,t}^{1+\varphi}+\beta\theta_{w}\left(\frac{\tilde{W}_{H,t}}{W_{H,t+1}}\right)^{-\varepsilon_{w}(1+\varphi)}N_{H,t+1}^{1+\varphi}+\left(\beta\theta_{w}\right)^{2}\left(\frac{\tilde{W}_{H,t}}{W_{H,t+2}}\right)^{-\varepsilon_{w}(1+\varphi)}N_{H,t+2}^{1+\varphi}+\cdots\right]\\
&=M^{w}\left[\begin{array}{l}\left(\frac{\tilde{W}_{H,t}}{W_{H,t}}\right)^{-\varepsilon_{w}(1+\varphi)}N_{H,t}^{1+\varphi}+\beta\theta_{w}\left(\frac{\tilde{W}_{H,t}}{W_{H,t}}\frac{W_{H,t}}{W_{H,t+1}}\right)^{-\varepsilon_{w}(1+\varphi)}N_{H,t+1}^{1+\varphi}\\+\left(\beta\theta_{w}\right)^{2}\left(\frac{\tilde{W}_{H,t}}{W_{H,t}}\frac{W_{H,t}}{W_{H,t+1}}\frac{W_{H,t+1}}{W_{H,t+2}}\right)^{-\varepsilon_{w}(1+\varphi)}N_{H,t+2}^{1+\varphi}+\cdots\end{array}\right]\\
&=M^{w}\left[\begin{array}{l}\left(\tilde{X}_{H,t}^{w}\right)^{-\varepsilon_{w}(1+\varphi)}N_{H,t}^{1+\varphi}+\beta\theta_{w}\left(\tilde{X}_{H,t}^{w}\right)^{-\varepsilon_{w}(1+\varphi)}\left(\Pi_{H,t+1}^{w}\right)^{\varepsilon_{w}(1+\varphi)}N_{H,t+1}^{1+\varphi}\\+\left(\beta\theta_{w}\right)^{2}\left(\tilde{X}_{H,t}^{w}\right)^{-\varepsilon_{w}(1+\varphi)}\left(\Pi_{H,t+1}^{w}\Pi_{H,t+2}^{w}\right)^{\varepsilon_{w}(1+\varphi)}N_{H,t+2}^{1+\varphi}+\cdots\end{array}\right]\end{aligned}$$

, which can be rewritten as:

$$K_{H,t}^{w}=M^{w}\sum_{k=0}^{\infty}\left(\beta\theta_{w}\right)^{k}\left(\tilde{X}_{H,t}^{w}\right)^{-\varepsilon_{w}(1+\varphi)}N_{H,t+k}^{1+\varphi}\left(\prod_{s=1}^{k}\Pi_{H,t+s}^{w}\right)^{\varepsilon_{w}(1+\varphi)}$$

Finally, second-order approximated wage equation is given by:

$$\bar{\nu}^{w}=\kappa_{w}\sum_{k=0}^{\infty}\beta^{k}\mathrm{E}_{t}\left[\begin{array}{l}\varphi n_{t+k}+c_{t+K}-x_{H,t+k}-w_{H,t+k}^{r}+\frac{\varphi(2+\varphi)}{2}n_{t+k}^{2}-\frac{1}{2}c_{t+k}^{2}\\-\frac{1}{2}x_{t+k}^{2}-\frac{1}{2}\left(w_{t+k}^{r}\right)^{2}+c_{t+k}n_{t+k}+c_{t+k}x_{H,t+k}+c_{t+k}w_{t+k}^{r}\\-n_{t+k}x_{H,t+k}-n_{t+k}w_{H,t+k}^{r}-x_{t+k}w_{H,t+k}^{r}+\frac{\varepsilon_{w}(1+\varphi)}{2}\left(\pi_{H,t+k}^{w}\right)^{2}\end{array}\right].\ (4\text{-}4\text{-}8)$$

4.5 Second-order Approximation of the Government Solvency Condition

4.5.1 Second-order Approximation of the Iterated Government Solvency Condition

Government solvency condition Eq.(1-1-65) is given by:

$$C_{t}^{-1}Z_{t}\frac{B_{t-1}^{n}}{P_{t}}=C_{t}^{-1}Z_{t}SP_{t}+\beta C_{t+1}^{-1}Z_{t+1}SP_{t+1}+\beta^{2}C_{t+2}^{-1}Z_{t+2}SP_{t+2}+\cdots,$$

which can be rewritten as:

$$C_{t}^{-1}Z_{t}B_{t-1}\Pi_{t}^{-1}=C_{t}^{-1}Z_{t}SP_{t}+\beta C_{t+1}^{-1}Z_{t+1}SP_{t+1}+\beta^{2}C_{t+2}^{-1}Z_{t+2}SP_{t+2}+\cdots.$$

Let define $W_t \equiv C_t^{-1} Z_t B_{t-1} \Pi_t^{-1}$. Then, the previous expression can be rewritten as:

$$\begin{aligned} W_t &= C_t^{-1} Z_t SP_t + \beta C_{t+1}^{-1} Z_{t+1} SP_{t+1} + \beta^2 C_{t+2}^{-1} Z_{t+2} SP_{t+2} + \cdots \\ &= C_t^{-1} Z_t SP_t + \left[\beta C_{t+1}^{-1} Z_{t+1} SP_{t+1} + \beta^2 C_{t+2}^{-1} Z_{t+2} SP_{t+2} + \cdots\right] \end{aligned}. \quad \text{(4-5-1)}$$

Forwarding Eq.(4-5-1) one period ahead yields:

$W_{t+1} = C_{t+1}^{-1} Z_{t+1} SP_{t+1} + \beta C_{t+2}^{-1} Z_{t+2} SP_{t+2} + \beta^2 C_{t+3}^{-1} Z_{t+3} SP_{t+3} + \cdots$.

Multiplying *β* on both sides of the previous expression yields:

$\beta W_{t+1} = \beta C_{t+1}^{-1} Z_{t+1} SP_{t+1} + \beta^2 C_{t+2}^{-1} Z_{t+2} SP_{t+2} + \beta^3 C_{t+3}^{-1} Z_{t+3} SP_{t+3} + \cdots$. (4-5-2)

Plugging Eq.(4-5-1) into Eq.(4-6-2) yields:

$W_t = C_t^{-1} Z_t SP_t + \beta W_{t+1}$. (4-5-3)

Let define $SP_t^{ITMC} \equiv C_t^{-1} Z_t SP_t$. Then the previous expression can be rewritten as:

$W_0 = SP_0^{ITMC} + \beta W_1$. (4-14-14)

First-order approximation of Eq.(4-14-14) yields:

$$\begin{aligned} W_0 &= W + f(W_0)_{SP_H^{ITMC}} \left(SP_{H,0}^{ITMC} - SP_H^{ITMC}\right) + f(W_0)_W (W_1 - W) + o\left(\|\xi\|^3\right) \\ &= W + SP_{H,0}^{ITMC} sp_{H,0}^{ITMC} + \beta W \omega_1 + \text{t.i.p.} + o\left(\|\xi\|^2\right) \end{aligned},$$

which can be rewritten as:

$$\frac{W_0 - W}{W} = \frac{SP_{H,0}^{ITMC}}{W} sp_{H,0}^{ITMC} + \beta \omega_1 + \text{t.i.p.} + o\left(\|\xi\|^2\right) \quad \text{(4-5-7)}$$

Eq.(4-14-14) implies:

$W(1-\beta) = SP^{ITMC}$

In the steady state. Plugging the previous expression into Eq.(4-5-7) yields:

$\omega_0 = (1-\beta) sp_0^{ITMC} + \beta \omega_1 + \text{t.i.p.} + o\left(\|\xi\|^2\right)$, (4-14-15)

with $\omega_0 \equiv \frac{W_0 - W}{W}$. Iterating forward Eq.(4-5-7) ahead yields:

$$\begin{aligned} \omega_0 &= (1-\beta) sp_0^{ITMC} + \beta \omega_1 + \text{t.i.p.} + o\left(\|\xi\|^2\right) \\ \omega_1 &= (1-\beta) sp_1^{ITMC} + \beta \omega_2 + \text{t.i.p.} + o\left(\|\xi\|^2\right) \\ &\vdots \\ \omega_j &= (1-\beta) sp_j^{ITMC} + \beta \omega_j + \text{t.i.p.} + o\left(\|\xi\|^2\right) \\ &\vdots \end{aligned}, \quad \text{(4-5-9)}$$

Combining Eqs.(4-5-7) and (4-14-15) yields:

$\bar{\omega}=(1-\beta)\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(sp_t^{ITMC}\right)+\text{t.i.p.}+o\left(\|\xi\|^2\right)$, (4-5-10)

where $\bar{\omega}\equiv\omega_0$.

4.5.2 Second-order approximation of the Fiscal Surplus in terms of Marginal Utility of Consumption

Second-order approximation of $SP_{H,t}^{ITMC}\equiv C_t^{-1}Z_t SP_{H,t}$ is given by:

$$sp_t^{ITMC}=-c_t+sp_t+\frac{1}{2}c_t^2-c_t sp_t-c_t z_t+sp_t z_t+o\left(\|\xi\|^3\right). \quad (4\text{-}5\text{-}11)$$

Plugging Eq.(4-5-10) into Eq.(4-5-9) yields:

$$\bar{\omega}=(1-\beta)\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(-c_t+sp_t+\frac{1}{2}c_t^2-c_t sp_t-c_t z_t+sp_t z_t\right)+\text{t.i.p.}+o\left(\|\xi\|^2\right). \quad (4\text{-}5\text{-}12)$$

4.5.3 Second-order Approximation of the Definition of Fiscal Surplus

The definition of fiscal surplus is given by:

$$SP_{H,t}\equiv X_{H,t}\left(\tau_{H,t}Y_{H,t}-G_{H,t}\right)-\zeta_{H,t}, \quad (4\text{-}5\text{-}13)$$

with $X_{H,t}\equiv P_{H,t}/P_t$.

Second-order approximation of Eq.(4-5-13) is given by:

$$\begin{aligned}SP_{H,t}&=SP_H+SP_{H\,X}\left(X_{H,t}-1\right)+SP_{H\,\tau}\left(\hat{\tau}_{H,t}-\tau_H\right)+SP_{H\,Y}\left(Y_{H,t}-Y_H\right)+SP_{H\,G_H}\left(G_{H,t}-G_H\right)\\&\quad+SP_{H\,\zeta_H}d\zeta_{H,t}+SP_{H\,X\tau}\left(X_{H,t}-1\right)\left(\hat{\tau}_{H,t}-\tau_H\right)+SP_{H\,XY}\left(X_{H,t}-1\right)\left(Y_{H,t}-Y_H\right)\\&\quad+SP_{H\,XG}\left(X_{H,t}-1\right)\left(G_{H,t}-G_H\right)+SP_{H\,\tau Y}\left(\hat{\tau}_{H,t}-\tau_H\right)\left(Y_{H,t}-Y_H\right)+o\left(\|\xi\|^3\right),\\&=SP_H+SP\left(x_{H,t}+\frac{1}{2}x_{H,t}^2\right)+Y\tau\left(\hat{\tau}_{H,t}+\frac{1}{2}\hat{\tau}_{H,t}^2\right)-Gg_{H,t}-d\zeta_{H,t}+Y\tau x_{H,t}\hat{\tau}_{H,t}+Y\tau x_{H,t}y_{H,t}\\&\quad-Gx_{H,t}g_{H,t}+\tau Y y_{H,t}\tau_{H,t}+o\left(\|\xi\|^3\right)\end{aligned}$$

which can be rewritten as:

$$\begin{aligned}sp_{H,t}&=x_{H,t}+\frac{1}{2}x_{H,t}^2+\frac{Y}{SP}\tau\left(\hat{\tau}_{H,t}+\frac{1}{2}\hat{\tau}_{H,t}^2\right)+\frac{\tau}{SP}Y\left(y_{H,t}+\frac{1}{2}y_{H,t}^2\right)-\frac{Y}{SP}\frac{G}{Y}g_{H,t}-\frac{Y}{SP}\frac{d\zeta_{H,t}}{Y_H},\\&\quad+\frac{Y}{SP}\tau x_{H,t}\hat{\tau}_{H,t}+\frac{Y}{SP}\tau x_{H,t}y_{H,t}-\frac{Y}{SP}\frac{G}{Y}x_{H,t}g_{H,t}+\frac{Y}{SP}\tau y_{H,t}\tau_{H,t}+o\left(\|\xi\|^3\right)\end{aligned}$$

with $sp_{H,t}\equiv\frac{SP_{H,t}-SP_H}{SP_H}$. Plugging $SP=(1-\beta)B$ into the previous expression yields:

$$sp_{H,t}=x_{H,t}+\frac{1}{2}x_{H,t}^{2}+\frac{Y}{(1-\beta)B}\tau\left(\hat{\tau}_{H,t}+\frac{1}{2}\hat{\tau}_{H,t}^{2}\right)+\frac{Y}{(1-\beta)B}\tau\left(y_{H,t}+\frac{1}{2}y_{H,t}^{2}\right)-\frac{Y}{(1-\beta)B}\frac{G}{Y}g_{H,t}$$
$$-\frac{Y}{(1-\beta)B}\frac{d\zeta_{H,t}}{Y_H}+\frac{Y}{(1-\beta)B}\tau x_{H,t}\hat{\tau}_{H,t}+\frac{Y}{(1-\beta)B}\tau x_{H,t}y_{H,t}-\frac{Y}{(1-\beta)B}\frac{G}{Y}x_{H,t}g_{H,t}$$
$$+\frac{Y}{(1-\beta)B}\tau y_{H,t}\tau_{H,t}+o\left(\|\xi\|^{3}\right).$$

Plugging $\sigma_B\equiv B/Y$ and $\sigma_G\equiv G/Y$ into the previous expression yields:

$$sp_{t}=x_{H,t}+\frac{\tau}{(1-\beta)\sigma_B}\hat{\tau}_t+\frac{\tau}{(1-\beta)\sigma_B}y_t-\frac{\sigma_G}{(1-\beta)\sigma_B}g_t-\frac{1}{(1-\beta)\sigma_B}\hat{\zeta}_t+\frac{1}{2}x_{H,t}^{2}$$
$$+\frac{\tau}{2(1-\beta)\sigma_B}\hat{\tau}_t^{2}+\frac{\tau}{2(1-\beta)\sigma_B}y_t^{2}+\frac{\tau}{(1-\beta)\sigma_B}x_{H,t}\hat{\tau}_t+\frac{\tau}{(1-\beta)\sigma_B}x_{H,t}y_t \quad ,(4\text{-}5\text{-}14)$$
$$-\frac{\sigma_G}{(1-\beta)\sigma_B}x_{H,t}g_t+\frac{\tau}{(1-\beta)\sigma_B}y_t\tau_t+o\left(\|\xi\|^{3}\right)$$

with $\hat{\zeta}_{H,t}\equiv\frac{d\zeta_{H,t}}{Y_H}$.

Multiplying c_t on both sides of Eq.(4-5-14) yields:

$$c_t sp_t=c_t x_{H,t}+\frac{\tau}{(1-\beta)\sigma_B}c_t\hat{\tau}_t+\frac{\tau}{(1-\beta)\sigma_B}c_t y_t-\frac{\sigma_G}{(1-\beta)\sigma_B}c_t g_t-\frac{1}{(1-\beta)\sigma_B}c_t\hat{\zeta}_t \quad .(4\text{-}5\text{-}15)$$
$$+o\left(\|\xi\|^{3}\right)$$

Multiplying $d_{H,t}$ on both sides of Eq.(4-5-14) yields:

$$sp_t z_t=x_{H,t}z_t+\frac{\tau}{(1-\beta)\sigma_B}\hat{\tau}_t z_t+\frac{\tau}{(1-\beta)\sigma_B}y_t z_t+\text{s.o.t.i.p.}+o\left(\|\xi\|^{3}\right)$$

Plugging Eqs.(4-5-14) and (4-5-15) and the previous exprssion into Eq.(4-5-12) yields:

$$\bar{\omega}=(1-\beta)\sum_{t=0}^{\infty}\beta^{t}\mathrm{E}_0\left(\begin{array}{l}-c_t+x_{H,t}+\frac{\tau}{(1-\beta)\sigma_B}\hat{\tau}_t+\frac{\tau}{(1-\beta)\sigma_B}y_t+\frac{1}{2}c_t^{2}+\frac{1}{2}x_{H,t}^{2}\\+\frac{\tau}{2(1-\beta)\sigma_B}\hat{\tau}_t^{2}+\frac{\tau}{2(1-\beta)\sigma_B}y_t^{2}-c_t x_{H,t}-\frac{\tau}{(1-\beta)\sigma_B}c_t\hat{\tau}_t\\-\frac{\tau}{(1-\beta)\sigma_B}c_t y_t+\frac{\sigma_G}{(1-\beta)\sigma_B}c_t g_t+\frac{1}{(1-\beta)\sigma_B}c_t\hat{\zeta}_t-c_t z_t\\+\frac{\tau}{(1-\beta)\sigma_B}x_{H,t}\hat{\tau}_t+\frac{\tau}{(1-\beta)\sigma_B}x_{H,t}y_t-\frac{\sigma_G}{(1-\beta)\sigma_B}x_{H,t}g_t\\+x_{H,t}z_t+\frac{\tau}{(1-\beta)\sigma_B}\hat{\tau}_t z_t+\frac{\tau}{(1-\beta)\sigma_B}y_t\tau_t+\frac{\tau}{(1-\beta)\sigma_B}y_t z_t\end{array}\right)+\text{t.i.p.}+o\left(\|\xi\|^{2}\right)$$

.(4-5-16)

4.7 Second-order Approximation of the Market Clearing Condition

Eq.(1-2-76) can be rewritten as:

$Y_t = (1-\upsilon) X_{H,t}^{-\eta} C_t + \upsilon S_t^{\eta} Z_{1,t}^{*} + G_t$. (4-7-1)

Where we use $X_{H,t} \equiv P_{H,t}/P_t$ and Eq.(1-1-20).

Second-order approximation of Eq.(4-7-1) is given by:

$$
\begin{aligned}
Y_H\begin{pmatrix} X_{H,t}, C_t, S_t, \\ Z_{1,t}^{*}, G_t \end{pmatrix} &= Y + Y_X\left(X_{H,t}-1\right) + Y_C\left(C_t - C\right) + Y_S\left(S_t - 1\right) + \frac{1}{2}Y_{XX}\left(X_{H,t}-1\right)^2 \\
&+ \frac{1}{2}Y_{SS}\left(S_t-1\right)^2 + Y_{XC}\left(X_{H,t}-1\right)\left(C_t - C\right) + Y_{SZ}\left(S_t-1\right)\left(Z_{1,t}^{*}-1\right) \\
&+ \text{t.i.p.} + o\left(\left\|\xi\right\|^3\right) \\
&= Y - \eta(1-\upsilon)C\left(x_{H,t} + \frac{1}{2}x_{H,t}^2\right) + (1-\upsilon)C\left(c_t + \frac{1}{2}c_t^2\right) + \upsilon\eta Z_1^{*}\left(s_t + \frac{1}{2}s_t^2\right) \\
&+ \frac{1}{2}\eta(\eta+1)(1-\upsilon)Cx_{H,t}^2 + \frac{1}{2}\upsilon\eta(\eta-1)Z_1^{*}s_t^2 - \eta(1-\upsilon)Cx_{H,t}c_t \\
&+ \upsilon\eta Z_1^{*}s_t z_{1,t}^{*} + \text{t.i.p.} + o\left(\left\|\xi\right\|^3\right)
\end{aligned},
$$

which can be rewritten as:

$$
\begin{aligned}
\frac{Y_t - Y}{Y} &= -\eta(1-\upsilon)\frac{C}{Y}\left(x_{H,t} + \frac{1}{2}x_{H,t}^2\right) + (1-\upsilon)\frac{C}{Y}\left(c_t + \frac{1}{2}c_t^2\right) + \upsilon\eta\frac{C}{Y}\left(s_t + \frac{1}{2}s_t^2\right) \\
&+ \frac{1}{2}\eta(\eta+1)(1-\upsilon)\frac{C}{Y}x_{H,t}^2 + \frac{1}{2}\upsilon\eta(\eta-1)\frac{C}{Y}s_t^2 - \eta(1-\upsilon)\frac{C}{Y}x_{H,t}c_t \\
&+ \upsilon\eta\frac{C}{Y}s_t z_{1,t}^{*} + \text{t.i.p.} + o\left(\left\|\xi\right\|^3\right) \\
&= -\eta(1-\upsilon)\frac{C}{Y}x_{H,t} + (1-\upsilon)\frac{C}{Y}c_t + \upsilon\eta\frac{C}{Y}s_t - \frac{\eta(1-\upsilon)}{2}\frac{C}{Y}\left[1-(\eta+1)\right]x_{H,t}^2 \\
&+ \frac{1-\upsilon}{2}\frac{C}{Y}c_t^2 + \frac{\upsilon\eta}{2}\frac{C}{Y}\left[1+(\eta-1)\right]s_t^2 - \eta(1-\upsilon)\frac{C}{Y}x_{H,t}c_t \\
&+ \upsilon\eta\frac{C}{Y}s_t z_{1,t}^{*} + \text{t.i.p.} + o\left(\left\|\xi\right\|^3\right) \\
&= -\eta(1-\upsilon)\frac{C}{Y}x_{H,t} + (1-\upsilon)\frac{C}{Y}c_t + \upsilon\eta\frac{C}{Y}s_t + \frac{\eta^2(1-\upsilon)}{2}\frac{C}{Y}x_{H,t}^2 \\
&+ \frac{1-\upsilon}{2}\frac{C}{Y}c_t^2 + \frac{\upsilon\eta^2}{2}\frac{C}{Y}s_t^2 - \eta(1-\upsilon)\frac{C}{Y}x_{H,t}c_t + \upsilon\eta\frac{C}{Y}s_t z_{1,t}^{*} \\
&+ \text{t.i.p.} + o\left(\left\|\xi\right\|^3\right)
\end{aligned}.
$$

Then, we have:

$$y_t = -\eta(1-\upsilon)\sigma_C x_{H,t} + (1-\upsilon)\sigma_C c_t + \upsilon\eta\sigma_C s_t + \frac{\eta^2(1-\upsilon)\sigma_C}{2} x_{H,t}^2 + \frac{(1-\upsilon)\sigma_C}{2} c_t^2$$
$$+ \frac{\upsilon\eta^2\sigma_C}{2} s_t^2 - \eta(1-\upsilon)\sigma_C x_{H,t} c_t + \upsilon\eta\sigma_C s_t z_{1,t}^* + \text{t.i.p.} + o\left(\|\xi\|^3\right).$$

Iterating the previous expression yields:

$$0 = \sum_{t=0}^{\infty} \beta^t \mathrm{E}_0 \left\{ \begin{array}{l} -y_t - \eta(1-\upsilon)\sigma_C x_{H,t} + (1-\upsilon)\sigma_C c_t + \upsilon\eta\sigma_C s_t \\ + \frac{\eta^2(1-\upsilon)\sigma_C}{2} x_{H,t}^2 + \frac{(1-\upsilon)\sigma_C}{2} c_t^2 \\ + \frac{\upsilon\eta^2\sigma_C}{2} s_t^2 - \eta(1-\upsilon)\sigma_C x_{H,t} c_t + \upsilon\eta\sigma_C s_t z_{1,t}^* \end{array} \right\} + \text{t.i.p.} + o\left(\|\xi\|^3\right). \quad (4\text{-}7\text{-}2)$$

4.8 Second-order Approximation of the Definition of the Relative Price

Definition of the relative price of PPI in the SOE is given by:

$$X_{H,t} \equiv \frac{P_{H,t}}{P_t},$$

which can be rewritten as:

$$\begin{aligned} X_{H,t} &\equiv \frac{P_{H,t}}{P_t} \\ &= \frac{1}{P_t} \frac{P_{H,t}}{P_{F,t}} P_{F,t} \\ &= \frac{1}{P_t} \frac{P_{F,t}}{S_t} \end{aligned}.$$

By raising both sides on the previous expression the $\eta - 1$ th power yields:

$$
\begin{aligned}
X_{H,t}^{\eta-1} &= \left(\frac{1}{P_t}\frac{P_{F,t}}{S_t}\right)^{\eta-1} \\
&= P_t^{1-\eta} P_{F,t}^{\eta-1} S_t^{1-\eta} \\
&= \left\{\left[(1-\upsilon)P_{H,t}^{1-\eta} + \upsilon P_{F,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}\right\}^{1-\eta} P_{F,t}^{\eta-1} S_t^{1-\eta} \\
&= \left[(1-\upsilon)P_{H,t}^{1-\eta} + \upsilon P_{F,t}^{1-\eta}\right] P_{F,t}^{\eta-1} S_t^{1-\eta} \\
&= (1-\upsilon)P_{H,t}^{1-\eta} P_{F,t}^{\eta-1} S_t^{1-\eta} + \upsilon P_{F,t}^{1-\eta} P_{F,t}^{\eta-1} S_t^{1-\eta} \\
&= (1-\upsilon)\left(\frac{P_{F,t}}{P_{H,t}}\right)^{\eta-1} S_t^{-(\eta-1)} + \upsilon P_{F,t}^{1-\eta} P_{F,t}^{-(\eta-1)} S_t^{1-\eta} \\
&= (1-\upsilon) S_t^{\eta-1} S_t^{-(\eta-1)} + \upsilon S_t^{1-\eta} \\
&= (1-\upsilon) + \upsilon S_t^{1-\eta}
\end{aligned}
.
$$

By raising on the both sides of the previous expression $1/(\eta-1)$ th power yields:

$$
X_{H,t} = \left[(1-\upsilon) + \upsilon S_t^{1-\eta}\right]^{\frac{1}{\eta-1}}. \quad \text{(4-8-1)}
$$

Second-order approximation of Eq.(4-14-24) is given by:

$$
\begin{aligned}
X_{H,t} &= 1 + X_{H\,S}(S_t - 1) + \frac{1}{2} X_{H\,SS}(S_t - 1)^2 + o\left(\|\xi\|^3\right) \\
&= 1 + X_{H\,S}\left(s_t + \frac{1}{2}s_t^2\right) + \frac{1}{2} X_{H\,SS} s_t^2 + o\left(\|\xi\|^3\right) \\
&= 1 + X_{H\,S} s_t + \frac{X_{H\,S} + X_{H\,SS}}{2} s_t^2 + o\left(\|\xi\|^3\right)
\end{aligned}
\quad . \text{ (4-8-2)}
$$

$X_{H\,S}$ and $X_{H\,SS}$ are given by:

$$
\begin{aligned}
X_{H\,S} &= \frac{1}{\eta-1}\left[(1-\upsilon) + \upsilon S^{-(\eta-1)}\right]^{\frac{1}{\eta-1}-1} \upsilon\left[-(\eta-1)\right] S^{-\eta} \\
&= \frac{1}{\eta-1}\left[(1-\upsilon) + \upsilon\right]^{\frac{1}{\eta-1}-1} \upsilon\left[-(\eta-1)\right] \\
&= \frac{\upsilon\left[-(\eta-1)\right]}{\eta-1} \\
&= -\upsilon
\end{aligned}
,
$$

$$
\begin{aligned}
X_{H\,SS} &= \frac{1}{\eta-1}\left(\frac{1}{\eta-1}-1\right)\left[(1-\upsilon)+\upsilon S^{-(\eta-1)}\right]^{\frac{1}{\eta-1}-2}\upsilon\left[-(\eta-1)\right]S^{-\eta}\upsilon\left[-(\eta-1)\right]S^{-\eta} \\
&\quad+\frac{1}{\eta-1}\left[(1-\alpha)+\upsilon S^{-(\eta-1)}\right]^{\frac{1}{\eta-1}-1}\upsilon(-\eta)\left[-(\eta-1)\right]S^{-\eta-1} \\
&=\frac{1}{\eta-1}\frac{1-(\eta-1)}{\eta-1}\left[(1-\upsilon)+\upsilon\right]^{\frac{1}{\eta-1}-2}\upsilon\left[-(\eta-1)\right]\upsilon\left[-(\eta-1)\right] \\
&\quad+\frac{1}{\eta-1}\left[(1-\upsilon)+\upsilon\right]^{\frac{1}{\eta-1}-1}\upsilon\eta(\eta-1) \\
&=\frac{-\eta}{(\eta-1)^2}\upsilon^2(\eta-1)^2+\frac{1}{\eta-1}\upsilon\eta(\eta-1) \\
&=-\eta\upsilon^2+\upsilon\eta \\
&=\upsilon\eta(1-\upsilon)
\end{aligned}
.
$$

Plugging those expression into Eq.(4-8-2) yields:

$$
\begin{aligned}
x_{H,t} &= -\upsilon s_t+\frac{(-\upsilon)+\upsilon\eta(1-\upsilon)}{2}s_t^2+o\left(\|\xi\|^3\right) \\
&= -\upsilon s_t+\frac{\upsilon\eta(1-\upsilon)-\upsilon}{2}s_t^2+o\left(\|\xi\|^3\right) \\
&= -\upsilon s_t+\frac{\upsilon\left[\eta(1-\upsilon)-1\right]}{2}s_t^2+o\left(\|\xi\|^3\right)
\end{aligned}
.
$$

Iterating the previous expression yields:

$$
0=\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{-x_{H,t}-\upsilon s_t+\frac{\upsilon\left[\eta(1-\upsilon)-1\right]}{2}s_t^2\right\}+o\left(\|\xi\|^3\right). \quad \text{(4-8-3)}
$$

4.9 Second-order Approximation of the International Risk Sharing Condition

Eq.(1-1-21) can be rewritten as:

$$
Q_t=C_t\frac{Z_{2,t}^*}{Z_t}. \quad \text{(4-9-1)}
$$

Second-order approximation of Eq.(4-9-1) is given by:

$$
\begin{aligned}
Q_t &= f\left(C_t, Z_t, Z_{2,t}^*\right) \\
&= 1 + f_C\left(C_t - C\right) + f_{CZ_2^*}\left(C_t - C\right)\left(Z_{2,t}^* - 1\right) + f_{CZ}\left(C_t - C\right)\left(Z_t - 1\right) \\
&\quad + \text{t.i.p.} + o\left(\|\xi\|^3\right) \\
&= 1 + C\left(c_t + \frac{1}{2}c_t^2\right) + Cc_t z_{2,t}^* - Cc_t z_t + \text{t.i.p.} + o\left(\|\xi\|^3\right) \\
&= 1 + Cc_t + C\frac{1}{2}c_t^2 + Cc_t z_{2,t}^* - Cc_t z_t + \text{t.i.p.} + o\left(\|\xi\|^3\right)
\end{aligned},
$$

which can be rewritten as:

$$
q_t = c_t + \frac{1}{2}c_t^2 + c_t z_{2,t}^* - c_t z_t + \text{t.i.p.} + o\left(\|\xi\|^3\right)
$$

By Plugging Eq.(3-1-4) into the previous expression, we have:

$$
(1-\alpha)s_t = c_t + \frac{1}{2}c_t^2 + c_t z_{2,t}^* - c_t z_t + \text{t.i.p.} + o\left(\|\xi\|^3\right). \quad (4\text{-}9\text{-}2)
$$

In the first order, Eq.(4-9-2) can be rewritten as:

$$
(1-\alpha)s_t = c_t + \text{t.i.p.} + o\left(\|\xi\|^2\right)
$$

Iterating the previous expression can be rewritten as:

$$
0 = \sum_{t=0}^{\infty} \beta^t \mathrm{E}_0\left[c_t - (1-\alpha)s_t\right] + \text{t.i.p.} + o\left(\|\xi\|^2\right). \quad (4\text{-}9\text{-}3)
$$

4.12 Second-order Approximation of the Production Function

Eq.(1-1-80) can be rewritten as:

$$
N_t = \frac{Y_t \Delta_t^p}{A_t}
$$

Second-order Approximation of the previous expression is given by:

$$
\begin{aligned}
N_t &= f\left(Y_t, \Delta_t^p, A_t\right) \\
&= N + f_y\left(Y_t - Y\right) + f_\Delta\left(\Delta_t^p - 1\right) + f_{YA}\left(Y_t - Y\right)\left(A_t - 1\right) + \text{t.i.p.} + o\left(\|\xi\|^2\right) \\
&= N + Y\left(y_t + \frac{1}{2}y_t^2\right) + Y\ln\Delta_t^p - Yy_t a_t \\
&\quad + \text{t.i.p.} + o\left(\|\xi\|^2\right)
\end{aligned},
$$

which can be rewritten as:

$$\frac{N_t - N}{N} = Y^{-1}\left[Y\left(y_t + \frac{1}{2}y_t^2\right) + Y\ln\Delta_t^p - Yy_t a_t\right]$$
$$+\text{t.i.p.} + o\left(\|\xi\|^2\right)$$
$$= \left(y_t + \frac{1}{2}y_t^2\right) + \ln\Delta_t^p - y_t a_t + \text{t.i.p.} + o\left(\|\xi\|^2\right)$$

Further:

$$n_t = y_t + \frac{1}{2}y_t^2 + \ln\Delta_t^p - y_t a_t + \text{t.i.p.} + o\left(\|\xi\|^2\right)$$

Iterating the previous expression yields:

$$0 = \sum_{t=0}^{\infty}\beta^t\left(-n_t + y_t + \frac{1}{2}y_t^2 + \frac{\varepsilon_p}{2\kappa_p}\pi_{H,t}^2 - y_t a_t\right). \text{ (4-12-1)}$$

4.13 Eliminating the Linear-terms

4.13.1 Undetermined Coefficients

In the first-order, Eqs.(4-2-17), (4-5-16), (4-3-1), (4-7-2), (4-8-3), (4-9-3), (4-4-8) and (4-12-1) are given by:

$$\tilde{W} = \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\Phi}{(1-\alpha)\sigma_C}y_t - \frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\} + o\left(\|\xi\|^3\right), \text{ (4-13-1)} \leftarrow\text{(4-2-17)}$$

① $$\bar{\omega} = (1-\beta)\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\begin{array}{l} -c_t + x_{H,t} + \dfrac{\tau}{(1-\beta)\sigma_B}\hat{\tau}_t \\ + \dfrac{\tau}{(1-\beta)\sigma_B}y_t \end{array}\right) + \text{t.i.p.} + o\left(\|\xi\|^2\right), \text{ (4-13-2)} \leftarrow\text{(4-5-16)}$$

② $$\bar{\nu}_H = \kappa\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\begin{array}{l} w_t^r - \dfrac{2\tau}{1-\tau}\hat{\tau}_t + \dfrac{\tau}{1-\tau}c_t - \dfrac{\tau}{1-\tau}y_t \\ -\dfrac{\tau}{1-\tau}x_{H,t} \end{array}\right) + \text{t.i.p.} + o\left(\|\xi\|^2\right), \text{ (4-13-3)} \leftarrow\text{(4-3-1)}$$

③ $$0 = \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left[\begin{array}{l} -y_t + (1-\upsilon)\sigma_C c_t - \eta(1-\upsilon)\sigma_C x_{H,t} \\ +\upsilon\eta\sigma_C s_t \end{array}\right] + \text{t.i.p.} + o\left(\|\xi\|^2\right) \text{ (4-13-4)} \leftarrow\text{(4-7-2)}$$

④ $$0 = \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(-x_{H,t} - \upsilon s_t\right) + o\left(\|\xi\|^2\right) \text{ (4-13-5)} \leftarrow\text{(4-8-3)}$$

⑤ $$0 = \sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left[c_t - (1-\upsilon)s_t\right] + \text{t.i.p.} + o\left(\|\xi\|^3\right) \text{ (4-13-6)} \leftarrow\text{(4-9-3)}$$

⑧ $\bar{\nu}^{w}=\kappa_{W}\sum_{t=0}^{\infty}\beta^{t}E_{0}\left(\varphi n_{t}+c_{t}-x_{H,t}-w_{t}^{r}\right)+\text{t.i.p.}+o\left(\|\xi\|^{2}\right)$, (4-13-9) ←(4-4-8)

⑨ $0=\sum_{t=0}^{\infty}\beta^{t}E_{0}\left(-n_{t}+y_{t}\right)$. (4-13-10) ←(4-12-1)

How Eq.(4-13-1) corresponds to Eqs.(4-13-2)--(4-13-6), (4-13-9) and (4-13-10). Let solve this problem by method of undetermined coefficients.

The system of coefficients of Eqs.(4-13-2)--(4-13-6), (4-13-9) and (4-13-10) is given by:

$$\begin{bmatrix} \dfrac{\Phi}{\sigma_C} \\ 0 \\ -\dfrac{\upsilon\eta(2-\upsilon)}{1-\upsilon} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{\tau}{(1-\beta)\sigma_B}\Theta_1-\dfrac{\tau}{1-\tau}\Theta_2-\Theta_3+\Theta_9 \\ \dfrac{\tau}{(1-\beta)\sigma_B}\Theta_1-\dfrac{2\tau}{1-\tau}\Theta_2 \\ \upsilon\eta\sigma_C\Theta_3-\upsilon\Theta_4-(1-\upsilon)\Theta_5 \\ \Theta_1-\dfrac{\tau}{1-\tau}\Theta_2-\eta(1-\upsilon)\sigma_C\Theta_3-\Theta_4-\Theta_8 \\ -\Theta_1+\dfrac{\tau}{1-\tau}\Theta_2+(1-\upsilon)\sigma_C\Theta_3+\Theta_5+\Theta_8 \\ \Theta_2-\Theta_8 \\ \varphi\Theta_8-\Theta_9 \end{bmatrix} \begin{matrix} y_{t,} \\ \hat{\tau}_t \\ s_t \\ x_{H,t} \\ c_t \\ w_{H,t}^{r} \\ n_{H,t} \end{matrix},$$

where the LHS is the vector of coefficients of $y_{H,t}$, $\hat{\tau}_{H,t}$, s_t, $x_{H,t}$, c_t, $w_{H,t}^{r}$ and $n_{H,t}$ in Eq.(4-14-1) and the Θ_1, $\Theta_2,\Theta_3,\Theta_4,\Theta_5$, Θ_8 and Θ_9 are undetermined coefficients on Eqs. (4-14-2)—(4-14-10), repectively. The previous expression can be rewritten as:

① $$\frac{\Phi}{\sigma_C}=\frac{\tau}{(1-\beta)\sigma_B}\Theta_1-\frac{\tau}{1-\tau}\Theta_2-\Theta_3+\Theta_9$$

② $$0=\frac{\tau}{(1-\beta)\sigma_B}\Theta_1-\frac{2\tau}{1-\tau}\Theta_2$$

③ $$\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}=-\upsilon\eta\sigma_C\Theta_3+\upsilon\Theta_4+(1-\upsilon)\Theta_5$$

④ $$0=\Theta_1-\frac{\tau}{1-\tau}\Theta_2-\eta(1-\upsilon)\sigma_C\Theta_3-\Theta_4-\Theta_8 \quad \text{(4-13-11)}$$

⑤ $$0=-\Theta_1+\frac{\tau}{1-\tau}\Theta_2+(1-\upsilon)\sigma_C\Theta_3+\Theta_5+\Theta_8$$

⑥ $$0=\Theta_2-\Theta_8$$

⑦ $$0=\varphi\Theta_8-\Theta_9$$

The 7th equality in Eq.(4-13-11) can be rewritten as:

$\Theta_9=\varphi\Theta_8$. (4-13-12)

The 6th equality in Eq.(4-14-11) can be rewritten as:

$\Theta_8 = \Theta_2$. (4-13-13)

The 5th equality in Eq.(4-13-11) can be rewritten as:

$$\Theta_5 = \Theta_1 - \frac{\tau}{1-\tau}\Theta_2 - (1-\upsilon)\sigma_C\Theta_3 - \Theta_8 . \text{ (4-13-16)}$$

The 4th equality in Eq.(4-13-11) can be rewritten as:

$$\Theta_4 = \Theta_1 - \frac{\tau}{1-\tau}\Theta_2 - \eta(1-\upsilon)\sigma_C\Theta_3 - \Theta_8 . \text{ (4-13-17)}$$

The 3rd equality in Eq.(4-13-11) can be rewritten as:

$$\Theta_3 = -\frac{2-\upsilon}{(1-\upsilon)\sigma_C} + \frac{1}{\eta\sigma_C}\Theta_4 + \frac{1-\upsilon}{\upsilon\eta\sigma_C}\Theta_5 . \text{ (4-13-18)}$$

The 2nd equality in Eq.(4-13-11) can be rewritten as:

$$\Theta_2 = \frac{1-\tau}{2(1-\beta)\sigma_B}\Theta_1 . \text{ (4-13-19)}$$

The 1st equality in Eq.(4-13-11) can be rewritten as:

$$\Theta_1 = \frac{(1-\beta)\sigma_B\Phi}{\tau\sigma_C} + \frac{(1-\beta)\sigma_B}{1-\tau}\Theta_2 + \frac{(1-\beta)\sigma_B}{\tau}\Theta_3 - \frac{(1-\beta)\sigma_B}{\tau}\Theta_9 . \text{ (4-13-20)}$$

Plugging Eqs.(4-13-19) into Eq. (4-13-20) yields:

$$\begin{aligned}\Theta_1 &= \frac{(1-\beta)\sigma_B\Phi}{\tau\sigma_C} + \frac{(1-\beta)\sigma_B}{1-\tau}\frac{1-\tau}{2(1-\beta)\sigma_B}\Theta_1 + \frac{(1-\beta)\sigma_B}{\tau}\Theta_3 - \frac{(1-\beta)\sigma_B}{\tau}\Theta_9 \\ &= \frac{(1-\beta)\sigma_B\Phi}{\tau\sigma_C} + \frac{1}{2}\Theta_1 + \frac{(1-\beta)\sigma_B}{\tau}\Theta_3 - \frac{(1-\beta)\sigma_B}{\tau}\Theta_9\end{aligned},$$

which can be rewritten as:

$$\frac{1}{2}\Theta_1 = \frac{(1-\beta)\sigma_B\Phi}{\tau\sigma_C} + \frac{(1-\beta)\sigma_B}{\tau}\Theta_3 - \frac{(1-\beta)\sigma_B}{\tau}\Theta_9 .$$

Then:

$$\Theta_1 = \frac{2(1-\beta)\sigma_B\Phi}{\tau\sigma_C} + \frac{2(1-\beta)\sigma_B}{\tau}\Theta_3 - \frac{2(1-\beta)\sigma_B}{\tau}\Theta_9 . \text{ (4-13-21)}$$

Plugging Eq.(4-13-18) into Eq.(4-13-21) yields:

$$\Theta_1 = \frac{2(1-\beta)\sigma_B\Phi}{\tau\sigma_C} + \frac{2(1-\beta)\sigma_B}{\tau}\left[-\frac{2-\upsilon}{(1-\upsilon)\sigma_C} + \frac{1}{\eta\sigma_C}\Theta_4 + \frac{1-\upsilon}{\upsilon\eta\sigma_C}\Theta_5\right] - \frac{2(1-\beta)\sigma_B}{\tau}\Theta_9$$

$$= \frac{2(1-\beta)\sigma_B\Phi}{\tau\sigma_C} - \frac{2(1-\beta)\sigma_B(2-\upsilon)}{\tau(1-\upsilon)\sigma_C} + \frac{2(1-\beta)\sigma_B}{\tau\eta\sigma_C}\Theta_4 + \frac{2(1-\beta)\sigma_B(1-\upsilon)}{\tau\upsilon\eta\sigma_C}\Theta_5 - \frac{2(1-\beta)\sigma_B}{\tau}\Theta_9$$

$$= \frac{2(1-\beta)\sigma_B[\Phi(1-\upsilon)-(2-\upsilon)]}{\tau(1-\upsilon)\sigma_C} + \frac{2(1-\beta)\sigma_B}{\tau\eta\sigma_C}\Theta_4 + \frac{2(1-\beta)\sigma_B(1-\upsilon)}{\tau\upsilon\eta\sigma_C}\Theta_5 - \frac{2(1-\beta)\sigma_B}{\tau}\Theta_9$$

. (4-13-22)

Plugging Eq.(4-13-17) into Eq.(4-13-22) yields:

$$\Theta_1 = \frac{2(1-\beta)\sigma_B[\Phi(1-\upsilon)-(2-\upsilon)]}{\tau(1-\upsilon)\sigma_C} + \frac{2(1-\beta)\sigma_B}{\tau\eta\sigma_C}\left[\Theta_1 - \frac{\tau}{1-\tau}\Theta_2 - \eta\sigma_C\Theta_3 - \Theta_8\right] + \frac{2(1-\beta)\sigma_B(1-\upsilon)}{\tau\upsilon\eta\sigma_C}\Theta_5 - \frac{2(1-\beta)\sigma_B}{\tau}\Theta_9$$

$$= \frac{2(1-\beta)\sigma_B[\Phi(1-\upsilon)-(2-\upsilon)]}{\tau(1-\upsilon)\sigma_C} + \frac{2(1-\beta)\sigma_B}{\tau\eta\sigma_C}\Theta_1 - \frac{2(1-\beta)\sigma_B}{\eta\sigma_C(1-\tau)}\Theta_2 - \frac{2(1-\beta)(1-\upsilon)\sigma_B}{\tau}\Theta_3 - \frac{2(1-\beta)\sigma_B}{\tau\eta\sigma_C}\Theta_8 + \frac{2(1-\beta)\sigma_B(1-\upsilon)}{\tau\upsilon\eta\sigma_C}\Theta_5 - \frac{2(1-\beta)\sigma_B}{\tau}\Theta_9$$

, which can be rewritten as:

$$\left[1 - \frac{2(1-\beta)\sigma_B}{\tau\eta\sigma_C}\right]\Theta_1 = \frac{2(1-\beta)\sigma_B[\Phi(1-\upsilon)-(2-\upsilon)]}{\tau(1-\upsilon)\sigma_C} - \frac{2(1-\beta)\sigma_B}{\eta\sigma_C(1-\tau)}\Theta_2 - \frac{2(1-\beta)(1-\upsilon)\sigma_B}{\tau}\Theta_3 - \frac{2(1-\beta)\sigma_B}{\tau\eta\sigma_C}\Theta_8 + \frac{2(1-\beta)\sigma_B(1-\upsilon)}{\tau\upsilon\eta\sigma_C}\Theta_5 - \frac{2(1-\beta)\sigma_B}{\tau}\Theta_9 .$$

Further,

$$\frac{\tau\eta\sigma_C - 2(1-\beta)\sigma_B}{\tau\eta\sigma_C}\Theta_1 = \frac{2(1-\beta)\sigma_B[\Phi(1-\upsilon)-(2-\upsilon)]}{\tau(1-\upsilon)\sigma_C} - \frac{2(1-\beta)\sigma_B}{\eta\sigma_C(1-\tau)}\Theta_2$$
$$-\frac{2(1-\beta)(1-\upsilon)\sigma_B}{\tau}\Theta_3 - \frac{2(1-\beta)\sigma_B}{\tau\eta\sigma_C}\Theta_8 + \frac{2(1-\beta)\sigma_B(1-\upsilon)}{\tau\upsilon\eta\sigma_C}\Theta_5 \cdot$$
$$-\frac{2(1-\beta)\sigma_B}{\tau}\Theta_9$$

Let define $\Gamma_1 \equiv \tau\eta\sigma_C$ and $\Gamma_2 \equiv \Gamma_1 - 2(1-\beta)\sigma_B$. Then, the previous expression can be rewritten as:

$$\frac{\Gamma_2}{\Gamma_1}\Theta_1 = \frac{(\Gamma_1-\Gamma_2)\eta[\Phi(1-\upsilon)-(2-\upsilon)]}{\Gamma_1(1-\upsilon)} - \frac{(\Gamma_1-\Gamma_2)\tau}{\Gamma_1(1-\tau)}\Theta_2$$
$$-\frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)}{\Gamma_1}\Theta_3 + \frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_1}\Theta_5 \cdot$$
$$-\frac{(\Gamma_1-\Gamma_2)}{\Gamma_1}\Theta_8 - \frac{\Gamma_1-\Gamma_2}{\tau}\Theta_9$$

Then:

$$\frac{\Gamma_2}{\Gamma_1}\Theta_1 = \frac{(\Gamma_1-\Gamma_2)\Gamma_3}{\Gamma_1} - \frac{(\Gamma_1-\Gamma_2)\tau}{\Gamma_1(1-\tau)}\Theta_2$$
$$-\frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)}{\Gamma_1}\Theta_3 + \frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_1}\Theta_5$$
$$-\frac{(\Gamma_1-\Gamma_2)}{\Gamma_1}\Theta_8 - \frac{\Gamma_1-\Gamma_2}{\tau}\Theta_9$$

with $\Gamma_3 \equiv \frac{\eta[\Phi(1-\upsilon)-(2-\upsilon)]}{(1-\upsilon)}$. Thus, we have:

$$\Theta_1 = \frac{(\Gamma_1-\Gamma_2)\Gamma_3}{\Gamma_2} - \frac{(\Gamma_1-\Gamma_2)\tau}{\Gamma_2(1-\tau)}\Theta_2 - \frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)}{\Gamma_2}\Theta_3 + \frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5$$
$$-\frac{(\Gamma_1-\Gamma_2)}{\Gamma_2}\Theta_8 - \frac{\Gamma_1(\Gamma_1-\Gamma_2)}{\tau\Gamma_2}\Theta_9 \cdot$$

Plugging Eqs.(4-13-12) and (4-13-13) into the previous expression yields:

$$\Theta_1 = \frac{(\Gamma_1 - \Gamma_2)\Gamma_3}{\Gamma_2} - \frac{(\Gamma_1 - \Gamma_2)\tau}{\Gamma_2(1-\tau)}\Theta_2 - \frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)}{\Gamma_2}\Theta_3$$
$$+\frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5 - \frac{(\Gamma_1-\Gamma_2)}{\Gamma_2}\Theta_8 - \frac{\varphi\Gamma_1(\Gamma_1-\Gamma_2)}{\tau\Gamma_2}\Theta_8$$
$$= \frac{(\Gamma_1 - \Gamma_2)\Gamma_3}{\Gamma_2} - \frac{(\Gamma_1 - \Gamma_2)\tau}{\Gamma_2(1-\tau)}\Theta_2 - \frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)}{\Gamma_2}\Theta_3$$
$$+\frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5 - \frac{(\Gamma_1-\Gamma_2)(\tau+\varphi)}{\tau\Gamma_2}\Theta_8$$
$$= \frac{(\Gamma_1 - \Gamma_2)\Gamma_3}{\Gamma_2} - \frac{(\Gamma_1 - \Gamma_2)\tau}{\Gamma_2(1-\tau)}\Theta_2 - \frac{(\Gamma_1-\Gamma_2)[\tau+\varphi]}{\tau\Gamma_2}\Theta_2$$
$$-\frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)}{\Gamma_2}\Theta_3 + \frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5$$
$$= \frac{(\Gamma_1 - \Gamma_2)\Gamma_3}{\Gamma_2} - \frac{(\Gamma_1-\Gamma_2)\{\tau^2+(1-\tau)[\tau+\varphi]\}}{\tau(1-\tau)\Gamma_2}\Theta_2$$
$$-\frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)}{\Gamma_2}\Theta_3 + \frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5 \quad . \text{(4-13-23)}$$

Eq. (4-13-21) can be rewritten as:

$$\Theta_3 = \frac{\tau}{2(1-\beta)\sigma_B}\left[\Theta_1 - \frac{2(1-\beta)\sigma_B\Phi}{\tau\sigma_C} + \frac{2(1-\beta)\sigma_B\varphi}{\tau}\Theta_2\right], \text{(4-13-24)}$$
$$= -\frac{\Phi}{\sigma_C} + \frac{\tau}{2(1-\beta)\sigma_B}\Theta_1 + \varphi\Theta_2$$

where we use Eqs. (4-13-12) and (4-13-13).

Plugging Eqs.(4-13-19) and (4-13-24) into Eq.(4-13-23) yields:

$$\Theta_1 = \frac{(\Gamma_1-\Gamma_2)\Gamma_3}{\Gamma_2} - \frac{(\Gamma_1-\Gamma_2)\{\tau^2+(1-\tau)[\tau+\varphi]\}}{\tau(1-\tau)\Gamma_2}\frac{1-\tau}{2(1-\beta)\sigma_B}\Theta_1$$

$$-\frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)}{\Gamma_2}\left[-\frac{\Phi}{\sigma_C}+\frac{\tau}{2(1-\beta)\sigma_B}\Theta_1+\varphi\frac{1-\tau}{2(1-\beta)\sigma_B}\Theta_1\right]$$

$$+\frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5$$

$$=\frac{(\Gamma_1-\Gamma_2)\Gamma_3}{\Gamma_2}+\frac{\eta(1-\upsilon)(\Gamma_1-\Gamma_2)\Phi}{\Gamma_2}$$

$$-\frac{(\Gamma_1-\Gamma_2)\{\tau^2+(1-\tau)[\tau+\varphi]\}}{\tau\Gamma_2 2(1-\beta)\sigma_B}\Theta_1-\frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)\tau}{\Gamma_2 2(1-\beta)\sigma_B}\Theta_1$$

$$-\frac{\eta\sigma_C(1-\upsilon)(\Gamma_1-\Gamma_2)\varphi(1-\tau)}{\Gamma_2 2(1-\beta)\sigma_B}\Theta_1+\frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5$$

$$=\frac{(\Gamma_1-\Gamma_2)\Gamma_3+\eta(1-\upsilon)(\Gamma_1-\Gamma_2)\Phi}{\Gamma_2}+\frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5$$

$$-\left\{\frac{\{\tau^2+(1-\tau)[\tau+\varphi]\}}{\tau\Gamma_2}+\frac{\eta\sigma_C(1-\upsilon)\tau}{\Gamma_2}+\frac{\eta\sigma_C(1-\upsilon)\varphi(1-\tau)}{\Gamma_2}\right\}\Theta_1$$

$$=\frac{(\Gamma_1-\Gamma_2)[\Gamma_3+\eta(1-\upsilon)\Phi]}{\Gamma_2}+\frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5$$

$$-\frac{\{\tau^2+(1-\tau)[\tau+\varphi]\}+\eta\sigma_C(1-\upsilon)\tau^2+\eta\sigma_C(1-\upsilon)\varphi(1-\tau)\tau}{\tau\Gamma_2}\Theta_1 ,$$

$$=\frac{(\Gamma_1-\Gamma_2)[\Gamma_3+\eta(1-\upsilon)\Phi]}{\Gamma_2}+\frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5$$

$$-\frac{\tau^2+(1-\tau)[\tau+\varphi]+\eta\sigma_C(1-\upsilon)\tau[\tau+\varphi(1-\tau)]}{\tau\Gamma_2}\Theta_1$$

which can be rewritten as:

$$\left(1+\frac{\Gamma_7}{\tau\Gamma_2}\right)\Theta_1=\frac{(\Gamma_1-\Gamma_2)[\Gamma_3+\eta(1-\upsilon)\Phi]}{\Gamma_2}+\frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5 .$$

Further:

$$\frac{\tau\Gamma_2+\Gamma_7}{\tau\Gamma_2}\Theta_1=\frac{(\Gamma_1-\Gamma_2)[\Gamma_3+\eta(1-\upsilon)\Phi]}{\Gamma_2}+\frac{(1-\upsilon)(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_2}\Theta_5$$

with $\Gamma_7\equiv\tau^2+(1-\tau)[\tau+\varphi]+\eta\sigma_C(1-\upsilon)\tau[\tau+\varphi(1-\tau)]$. Then:

$$\Theta_1 = \frac{\tau(\Gamma_1-\Gamma_2)[\Gamma_3+\eta(1-\upsilon)\Phi]}{\Gamma_8} + \frac{(1-\upsilon)\tau(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_8}\Theta_5, \quad (4\text{-}13\text{-}25)$$

with $\Gamma_8 \equiv \tau\Gamma_2 + \Gamma_7$.

Plugging Eq.(4-13-13), (4-13-19) and (4-13-24) into Eq.(4-13-16) yields:

$$\begin{aligned}
\Theta_5 &= \Theta_1 - \frac{\tau}{2(1-\beta)\sigma_B}\Theta_1 - (1-\upsilon)\sigma_C\left[-\frac{\Phi}{\sigma_C} + \frac{\tau}{2(1-\beta)\sigma_B}\Theta_1 + \frac{\varphi(1-\tau)}{2(1-\beta)\sigma_B}\Theta_1\right] \\
&\quad - \frac{1-\tau}{2(1-\beta)\sigma_B}\Theta_1 \\
&= (1-\upsilon)\Phi + \frac{2(1-\beta)\sigma_B - \tau}{2(1-\beta)\sigma_B}\Theta_1 - \frac{(1-\upsilon)\sigma_C[\tau-\varphi(1-\tau)]}{2(1-\beta)\sigma_B}\Theta_1 \\
&= (1-\upsilon)\Phi + \frac{2(1-\beta)\sigma_B - \tau - (1-\upsilon)\sigma_C[\tau-\varphi(1-\tau)]}{2(1-\beta)\sigma_B}\Theta_1
\end{aligned}$$

Plugging the previous expression into Eq.(4-13-25) yields:

$$\begin{aligned}
\Theta_1 &= \frac{\tau(\Gamma_1-\Gamma_2)[\Gamma_3+\eta(1-\upsilon)\Phi]}{\Gamma_8} \\
&\quad + \frac{(1-\upsilon)\tau(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_8}\left\{(1-\upsilon)\Phi + \frac{2(1-\beta)\sigma_B - \tau - (1-\upsilon)\sigma_C[\tau-\varphi(1-\tau)]}{2(1-\beta)\sigma_B}\Theta_1\right\} \\
&= \frac{\tau(\Gamma_1-\Gamma_2)[\Gamma_3+\eta(1-\upsilon)\Phi]}{\Gamma_8} + \frac{\tau(\Gamma_1-\Gamma_2)(1-\upsilon)^2\Phi}{\upsilon\Gamma_8} \\
&\quad + \frac{(1-\upsilon)\tau(\Gamma_1-\Gamma_2)}{\upsilon\Gamma_8}\frac{2(1-\beta)\sigma_B - \tau - (1-\upsilon)\sigma_C[\tau-\varphi(1-\tau)]}{2(1-\beta)\sigma_B}\Theta_1 \\
&= \frac{\tau(\Gamma_1-\Gamma_2)\left[\Gamma_3\upsilon + \Phi(1-\upsilon)\eta\upsilon + \Phi(1-\upsilon)^2\right]}{\upsilon\Gamma_8} \\
&\quad + \frac{(1-\upsilon)\tau\{2(1-\beta)\sigma_B - \tau - (1-\upsilon)\sigma_C[\tau-\varphi(1-\tau)]\}}{\upsilon\Gamma_8}\Theta_1 \\
&= \frac{\tau 2(1-\beta)\sigma_B\{\Gamma_3\upsilon + \Phi(1-\upsilon)[\eta\upsilon + (1-\upsilon)]\}}{\upsilon\Gamma_8} \\
&\quad + \frac{(1-\upsilon)\tau\{2(1-\beta)\sigma_B - \tau - (1-\upsilon)\sigma_C[\tau-\varphi(1-\tau)]\}}{\upsilon\Gamma_8}\Theta_1,
\end{aligned}$$

which can be rewritten as:

$$\left\{1-\frac{(1-\upsilon)\tau\left\{2(1-\beta)\sigma_B-\tau-(1-\upsilon)\sigma_C\left[\tau-\varphi(1-\tau)\right]\right\}}{\upsilon\Gamma_8}\right\}\Theta_1 = \frac{\tau 2(1-\beta)\sigma_B\left\{\Gamma_3\upsilon+\Phi(1-\upsilon)\left[1+\upsilon(\eta-1)\right]\right\}}{\upsilon\Gamma_8}.$$

Further:

$$\frac{\upsilon\Gamma_8-(1-\upsilon)\tau\left\{2(1-\beta)\sigma_B-\tau-(1-\upsilon)\sigma_C\left[\tau-\varphi(1-\tau)\right]\right\}}{\upsilon\Gamma_8}\Theta_1 = \frac{\tau 2(1-\beta)\sigma_B\left\{\Gamma_3\upsilon+\Phi(1-\upsilon)\left[1+\upsilon(\eta-1)\right]\right\}}{\upsilon\Gamma_8}.$$

Then, we have:

$$\left\{\upsilon\Gamma_8-(1-\upsilon)\tau\left\{2(1-\beta)\sigma_B-\tau-(1-\upsilon)\sigma_C\left[\tau-\varphi(1-\tau)\right]\right\}\right\}\Theta_1 = \tau 2(1-\beta)\sigma_B\left\{\Gamma_3\upsilon+\Phi(1-\upsilon)\left[1+\upsilon(\eta-1)\right]\right\}.$$

Let define $\Gamma_9 \equiv 2(1-\beta)\sigma_B-\tau-(1-\upsilon)\sigma_C\left[\tau-\varphi(1-\tau)\right]$ and $\Gamma_{10} \equiv \Gamma_3\upsilon+\Phi(1-\upsilon)\left[1+\upsilon(\eta-1)\right]$.

Then, the previous expression can be rewritten as:

$$\Theta_1 = \frac{\tau 2(1-\beta)\sigma_B\Gamma_{10}}{\upsilon\Gamma_8-(1-\upsilon)\tau\Gamma_9}. \quad (4\text{-}13\text{-}26)$$

Plugging Eq.(4-13-13) into Eq.(4-13-17) yields:

$$\begin{aligned}\Theta_4 &= \Theta_1-\frac{\tau}{1-\tau}\Theta_2-\eta(1-\upsilon)\sigma_C\Theta_3-\Theta_2 \\ &= \Theta_1-\frac{1}{1-\tau}\Theta_2-\eta(1-\upsilon)\sigma_C\Theta_3\end{aligned}. \quad (4\text{-}13\text{-}27)$$

Plugging Eq.(4-14-13) into Eq.(4-14-16) yields:

$$\Theta_5 = \Theta_1-\frac{1}{1-\tau}\Theta_2-(1-\upsilon)\sigma_C\Theta_3. \quad (4\text{-}13\text{-}28)$$

Note that:

Θ_1 is given by Eq.(4-13-26), Θ_2 is given by Eq.(4-13-19), Θ_3 is given by Eq.(4-13-24), Θ_4 is given by Eq.(4-13-27), Θ_5 is given by Eq.(4-13-28), Θ_8 is given by Eq.(4-13-13) and Θ_9 is given by Eq.(4-13-12).

Then, the solution of Eq.(4-13-1) is given by:

$$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\Phi}{\sigma_C}y_t-\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}$$

$$=\Theta_1(1-\beta)^{-1}\varpi_H+\Theta_2\kappa^{-1}\overline{\nu}_H+\Theta_8\kappa_W^{-1}\overline{\nu}_H^W$$

$$+\Theta_1\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\begin{aligned}&\frac{1}{2}x_{H,t}^2+\frac{\tau}{2(1-\beta)\sigma_B}\hat{\tau}_t^2+\frac{\tau}{2(1-\beta)\sigma_B}y_t^2+\frac{\tau}{(1-\beta)\sigma_B}x_{H,t}\hat{\tau}_t\\&+\frac{\tau}{(1-\beta)\sigma_B}x_{H,t}y_t-\frac{\sigma_G}{(1-\beta)\sigma_B}x_{H,t}g_t+\frac{\tau}{(1-\beta)\sigma_B}y_t\tau_t\\&+\frac{1}{2}c_t^2-c_tx_{H,t}-\frac{\tau}{(1-\beta)\sigma_B}c_t\hat{\tau}_t-\frac{\tau}{(1-\beta)\sigma_B}c_ty_t\\&+\frac{\sigma_G}{(1-\beta)\sigma_B}c_tg_t+\frac{1}{(1-\beta)\sigma_B}c_t\hat{\zeta}_t-c_tz_t+x_{H,t}z_t\\&+\frac{\tau}{(1-\beta)\sigma_B}\hat{\tau}_tz_t+\frac{\tau}{(1-\beta)\sigma_B}y_tz_t\end{aligned}\right)$$

$$+\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{aligned}&\frac{1}{2}\left(w_t^r\right)^2+\frac{\tau}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{\tau}{2(1-\tau)}x_{H,t}^2-w_t^ra_t\\&-c_tw_t^r+c_ta_t+y_tw_t^r-y_ta_t+x_tw_{H,t}^r-x_{H,t}a_t+\frac{\varepsilon_p}{2}\pi_{H,t}^2\end{aligned}\right\}$$

$$+\Theta_3\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{aligned}&\frac{(1-\upsilon)\sigma_C}{2}c_t^2+\frac{\eta^2(1-\upsilon)\sigma_C}{2}x_{H,t}^2+\frac{\upsilon\eta^2\sigma_C}{2}s_t^2\\&-\eta(1-\upsilon)\sigma_Cx_{H,t}c_t+\upsilon\eta\sigma_Cs_tz_{1,t}^*\end{aligned}\right\}$$

$$+\Theta_4\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\upsilon\left[\eta(1-\upsilon)-1\right]}{2}s_t^2\right\}$$

$$+\Theta_8\sum_{t=0}^{\infty}\beta^k \mathrm{E}_0\left[\begin{aligned}&\frac{\varphi(2+\varphi)}{2}n_t^2-\frac{1}{2}c_t^2-\frac{1}{2}x_{H,t}^2-\frac{1}{2}\left(w_{t+k}^r\right)^2+c_tn_t+c_{t+k}x_{H,t}\\&+c_tw_t^r-n_tx_{H,t}-n_tw_t^r-x_{H,t}w_t^r+\frac{\varepsilon_W(1+\varphi)}{2}\left(\pi_{H,t}^W\right)^2\end{aligned}\right] \quad , (4\text{-}13\text{-}29)$$

$$+\Theta_9\sum_{t=0}^{\infty}\beta^t\left(\frac{1}{2}y_t^2+\frac{\varepsilon_p}{2\kappa_p}\pi_{H,t}^2-y_ta_t\right)+\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}$$

$$+o\left(\|\xi\|^2\right)$$

with:

$$\Upsilon_0\equiv\Theta_1(1-\beta)^{-1}\varpi_H+\Theta_2\kappa^{-1}\overline{\nu}_H+\Theta_8\kappa_W^{-1}\overline{\nu}_H^W .$$

Above results imply that Θ_6 must be disappear.

Now we focus on lines 4 and 7 in Eq.(4-13-29). Plugging Eq.(4-13-13) into lines 4 and 7 in Eq.(4-13-29) yields:

$$+\Theta_2\sum_{t=0}^{\infty}\beta^t E_0\left\{\begin{array}{l}\frac{1}{2}\left(w_t^r\right)^2+\frac{\tau}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{\tau}{2(1-\tau)}x_{H,t}^2-w_t^r a_t\\ -c_t w_t^r+c_t a_t+y_t w_t^r-y_t a_{H,t}+x_t w_{H,t}^r-x_{H,t}a_t+\frac{\varepsilon_p}{2}\pi_{H,t}^2\end{array}\right\}$$

$$+\Theta_8\sum_{t=0}^{\infty}\beta^k E_0\left[\begin{array}{l}\frac{\varphi(2+\varphi)}{2}n_t^2-\frac{1}{2}c_t^2-\frac{1}{2}x_{H,t}^2-\frac{1}{2}\left(w_{t+k}^r\right)^2+c_t n_t+c_{t+k}x_{H,t}\\ +c_t w_t^r-n_t x_{H,t}-n_t w_{H,t}^r-x_{H,t}w_{H,t}^r+\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{array}\right]$$

$$=+\Theta_2\sum_{t=0}^{\infty}\beta^t E_0\left\{\begin{array}{l}\frac{1}{2}\left(w_t^r\right)^2+\frac{\tau}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{\tau}{2(1-\tau)}x_{H,t}^2-w_t^r a_t\\ -c_t w_t^r+c_t a_t+y_t w_t^r-y_t a_{H,t}+x_t w_{H,t}^r-x_{H,t}a_t+\frac{\varepsilon_p}{2}\pi_{H,t}^2\\ +\frac{\varphi(2+\varphi)}{2}n_t^2-\frac{1}{2}c_t^2-\frac{1}{2}x_{H,t}^2-\frac{1}{2}\left(w_{t+k}^r\right)^2+c_t n_t+c_t x_{H,t}\\ +c_t w_t^r-n_t x_{H,t}-n_t w_t^r-x_{H,t}w_t^r+\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{array}\right\}$$

$$=+\Theta_2\sum_{t=0}^{\infty}\beta^t E_0\left\{\begin{array}{l}\left[\frac{\tau}{2(1-\tau)}-\frac{1}{2}\right]c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\left[\frac{\tau}{2(1-\tau)}+\frac{1}{2}\right]x_{H,t}^2+\frac{\varphi(2+\varphi)}{2}n_t^2\\ +c_t a_t-y_t a_{H,t}-x_{H,t}a_t+c_t n_t+c_t x_{H,t}-n_t x_{H,t}\\ +\left[y_t-\left(a_t+n_t\right)\right]w_t^r+\frac{\varepsilon_p}{2}\pi_{H,t}^2+\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{array}\right\}$$

$$=+\Theta_2\sum_{t=0}^{\infty}\beta^t E_0\left\{\begin{array}{l}\frac{\tau-(1-\tau)}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{1}{2(1-\tau)}x_t^2+\frac{\varphi(2+\varphi)}{2}n_t^2\\ +c_t a_t-y_t a_{H,t}-x_{H,t}a_t+c_t n_t+c_t x_{H,t}-n_t x_{H,t}+\frac{\varepsilon_p}{2}\pi_{H,t}^2\\ +\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2\end{array}\right\},$$

where we use Eq.(4-13-33). Note that cross term of $w_{H,t}^r$ no longer disappear as long as $\alpha\neq 0$. Plugging the previous expression into Eq.(4-13-29) yields:

$$
\begin{aligned}
&\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\Phi}{\sigma_C}y_t-\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}\\
&=\Theta_1(1-\beta)^{-1}\overline{\omega}_H+\Theta_2\kappa^{-1}\overline{\nu}_H+\Theta_8\kappa_W^{-1}\overline{\nu}_H^W\\
&+\Theta_1\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left(\begin{aligned}
&\frac{1}{2}x_{H,t}^2+\frac{\tau}{2(1-\beta)\sigma_B}\hat{\tau}_t^2+\frac{\tau}{2(1-\beta)\sigma_B}y_t^2+\frac{\tau}{(1-\beta)\sigma_B}x_{H,t}\hat{\tau}_t\\
&+\frac{\tau}{(1-\beta)\sigma_B}x_{H,t}y_t-\frac{\sigma_G}{(1-\beta)\sigma_B}x_{H,t}g_t+\frac{\tau}{(1-\beta)\sigma_B}y_t\tau_t\\
&+\frac{1}{2}c_t^2-c_tx_{H,t}-\frac{\tau}{(1-\beta)\sigma_B}c_t\hat{\tau}_t-\frac{\tau}{(1-\beta)\sigma_B}c_ty_t\\
&+\frac{\sigma_G}{(1-\beta)\sigma_B}c_tg_t+\frac{1}{(1-\beta)\sigma_B}c_t\hat{\zeta}_t-c_tz_t+x_{H,t}z_t\\
&+\frac{\tau}{(1-\beta)\sigma_B}\hat{\tau}_tz_t+\frac{\tau}{(1-\beta)\sigma_B}y_tz_t
\end{aligned}\right)\\
&+\Theta_2\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{\begin{aligned}
&\frac{\tau-(1-\tau)}{2(1-\tau)}c_t^2-\frac{\tau}{2(1-\tau)}y_t^2-\frac{1}{2(1-\tau)}x_t^2+\frac{\varphi(2+\varphi)}{2}n_t^2\\
&+c_ta_t-y_ta_{H,t}-x_{H,t}a_t+c_tn_t+c_tx_{H,t}-n_tx_{H,t}+\frac{\varepsilon_p}{2}\pi_{H,t}^2\\
&+\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2+\frac{\varphi}{2}y_t^2+\frac{\varphi\varepsilon_p}{2\kappa_p}\pi_{H,t}^2\\
&-\varphi y_ta_t
\end{aligned}\right.\\
&+\Theta_3\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{\begin{aligned}
&\frac{(1-\upsilon)\sigma_C}{2}c_t^2+\frac{\eta^2(1-\upsilon)\sigma_C}{2}x_{H,t}^2+\frac{\upsilon\eta^2\sigma_C}{2}s_t^2\\
&-\eta(1-\upsilon)\sigma_Cx_{H,t}c_t+\upsilon\eta\sigma_Cs_tz_{1,t}^*
\end{aligned}\right\}\\
&+\Theta_4\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{\frac{\upsilon[\eta(1-\upsilon)-1]}{2}s_t^2\right\}\\
&+\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}\\
&+o\left(\|\xi\|^2\right)
\end{aligned}
$$

where we use $\Theta_9=\varphi\Theta_8$. Eq.(4-13-12). The 4[th] line in the previous expression can be rewritten as:

$$+\Theta_2 \sum_{t=0}^{\infty} \beta^t \mathrm{E}_0 \left\{ \begin{array}{l} \frac{\tau-(1-\tau)}{2(1-\tau)} c_t^2 - \frac{\tau}{2(1-\tau)} y_t^2 - \frac{1}{2(1-\tau)} x_t^2 + \frac{\varphi(2+\varphi)}{2} n_t^2 \\ + c_t a_t - y_t a_{H,t} - x_{H,t} a_t + c_t n_t + c_t x_{H,t} - n_t x_{H,t} + \frac{\varepsilon_p}{2} \pi_{H,t}^2 \\ + \frac{\varepsilon_w (1+\varphi)}{2} \left(\pi_{H,t}^w\right)^2 + \frac{\varphi}{(1-\alpha)2} y_t^2 + \frac{\varphi \varepsilon_p}{(1-\alpha)2\kappa_p} \pi_{H,t}^2 \\ -\varphi y_t a_t \end{array} \right\} .$$

$$= +\Theta_2 \sum_{t=0}^{\infty} \beta^t \mathrm{E}_0 \left\{ \begin{array}{l} \frac{\tau-(1-\tau)}{2(1-\tau)} c_t^2 + \frac{\varphi(1-\tau)-\tau}{(1-\tau)2} y_t^2 - \frac{1}{2(1-\tau)} x_t^2 + \frac{\varphi(2+\varphi)}{2} n_t^2 \\ + c_t a_t - \frac{+\varphi}{1-\alpha} y_t a_t - x_{H,t} a_t + c_t n_t + c_t x_{H,t} - n_t x_{H,t} \\ + \frac{\varepsilon_p \left(\kappa_p + \varphi\right)}{2\kappa_p} \pi_{H,t}^2 + \frac{\varepsilon_w (1+\varphi)}{2} \left(\pi_{H,t}^w\right)^2 - \alpha n_t w_t^r \end{array} \right\}$$

Lastly, we have:

, (4-13-34)’

$$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\Phi}{\sigma_C}y_t-\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}$$

$$=\Theta_1(1-\beta)^{-1}\bar{\omega}_H+\Theta_2\kappa^{-1}\bar{\nu}_H+\Theta_8\kappa_W^{-1}\bar{\nu}_H^W$$

$$+\Theta_1\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{array}{l}\frac{1}{2}x_{H,t}^2+\frac{\tau}{2(1-\beta)\sigma_B}y_t^2\\ +\frac{\tau}{(1-\beta)\sigma_B}x_{H,t}y_t-\frac{\sigma_G}{(1-\beta)\sigma_B}x_{H,t}g_t\\ +\frac{1}{2}c_t^2-c_tx_{H,t}-\frac{\tau}{(1-\beta)\sigma_B}c_ty_t\\ +\frac{\sigma_G}{(1-\beta)\sigma_B}c_tg_t+\frac{1}{(1-\beta)\sigma_B}c_t\hat{\zeta}_t-c_tz_t+x_{H,t}z_t\\ +\frac{\tau}{(1-\beta)\sigma_B}y_tz_t\end{array}\right\}$$

$$+\Theta_2\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{array}{l}\frac{\tau-(1-\tau)}{2(1-\tau)}c_t^2+\frac{\varphi(1-\tau)-\tau}{(1-\tau)2}y_t^2-\frac{1}{2(1-\tau)}x_t^2\\ +\frac{\varphi(2+\varphi)}{2}n_t^2+c_ta_t-\varphi y_ta_t-x_{H,t}a_t+c_tn_t+c_tx_{H,t}\\ -n_tx_{H,t}+\frac{\varepsilon_p\left[\kappa_p+\varphi\right]}{2\kappa_p}\pi_{H,t}^2+\frac{\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2-\alpha n_tw_t^r\end{array}\right\}$$

$$+\Theta_3\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{array}{l}\frac{(1-\upsilon)\sigma_C}{2}c_t^2+\frac{\eta^2(1-\upsilon)\sigma_C}{2}x_{H,t}^2+\frac{\upsilon\eta^2\sigma_C}{2}s_t^2\\ -\eta(1-\upsilon)\sigma_Cx_{H,t}c_t+\upsilon\eta\sigma_Cs_tz_{1,t}^*\end{array}\right\}$$

$$+\Theta_4\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\upsilon\left[\eta(1-\upsilon)-1\right]}{2}s_t^2\right\}$$

$$+\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}$$

$$+o\left(\|\xi\|^2\right)$$

which can be rewritten as:

$$
\begin{aligned}
&\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\Phi}{\sigma_C}y_t-\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\} \\
&=\Theta_1(1-\beta)^{-1}\bar{\omega}_H+\Theta_2\kappa^{-1}\bar{\nu}_H+\Theta_8\kappa_W^{-1}\bar{\nu}_H^W \\
&+\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\begin{array}{l}
\left[\frac{\Theta_1}{2}-\frac{\Theta_2}{2(1-\tau)}+\frac{\Theta_3\eta^2(1-\upsilon)\sigma_C}{2}\right]x_{H,t}^2 \\
+\left[\frac{\Theta_1\tau}{2(1-\beta)\sigma_B}+\frac{\Theta_2[\varphi(1-\tau)-\tau]}{(1-\tau)2}\right]y_t^2 \\
+\left[\frac{\Theta_1}{2}+\frac{\tau-(1-\tau)\Theta_2}{2(1-\tau)}+\frac{\Theta_3(1-\upsilon)\sigma_C}{2}\right]c_t^2 \\
+\left[\frac{\Theta_3\upsilon\eta^2\sigma_C}{2}+\frac{\Theta_4\upsilon[\eta(1-\upsilon)-1]}{2}\right]s_t^2 \\
-[\Theta_1-\Theta_2+\Theta_3\eta(1-\upsilon)\sigma_C]c_t x_{H,t} \\
+\frac{\Theta_1\tau}{(1-\beta)\sigma_B}x_{H,t}y_t-\frac{\Theta_1\tau}{(1-\beta)\sigma_B}c_t y_t+\Theta_2 c_t n_t \\
-\Theta_2 n_t x_{H,t}-\frac{\Theta_1\sigma_G}{(1-\beta)\sigma_B}x_{H,t}g_t+\frac{\Theta_1\sigma_G}{(1-\beta)\sigma_B}c_t g_t \\
+\Theta_2 c_t a_t-\varphi\Theta_2 y_t a_t-\Theta_2 x_{H,t}a_t \\
+\frac{\Theta_1}{(1-\beta)\sigma_B}c_t\hat{\zeta}_t-\Theta_1 c_t z_t+\Theta_1 x_{H,t}z_t \\
+\frac{\Theta_1\tau}{(1-\beta)\sigma_B}y_t z_t+\Theta_3\upsilon\eta\sigma_C s_t z_{1,t}^* \\
+\frac{\Theta_2\varphi(2+\varphi)}{2}n_t^2-\alpha\Theta_2 n_t w_t^r \\
+\frac{\Theta_2\varepsilon_p[\kappa_p+\varphi]}{2\kappa_p}\pi_{H,t}^2+\frac{\Theta_2\varepsilon_w(1+\varphi)}{2}\left(\pi_{H,t}^w\right)^2
\end{array}\right) \\
&+\mathrm{T}_t+\Upsilon_0 \Big| +\text{t.i.p.} \\
&+o\left(\|\xi\|^2\right)
\end{aligned}
\quad (4\text{-}13\text{-}35)
$$

Eq.(3-1-8)’ can be rewritten as:

$$c_t=\frac{1}{(1-\upsilon)\sigma_C}y_t-\frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)}s_t-\frac{\upsilon}{(1-\upsilon)}z_{1,t}^*-\frac{\sigma_G}{(1-\upsilon)\sigma_C}g_t. \quad (4\text{-}13\text{-}30)'$$

Eq.(3-1-11) can be rewritten as:

$$s_t=\frac{1}{1-\upsilon}c_t-\frac{1}{1-\upsilon}z_t+\frac{1}{1-\upsilon}z_{2,t}^*. \quad (4\text{-}13\text{-}30)''$$

Eq.(3-1-26) can be rewritten as:

$y_t = (1-\alpha)n_t + a_t$. (4-13-30)''

Plugging Eq.(4-13-30)'' into Eq. (4-13-30)' yields:

$$c_t = \frac{1}{(1-\upsilon)\sigma_C} y_t - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)}\left(\frac{1}{1-\upsilon}c_t - \frac{1}{1-\upsilon}z_t + \frac{1}{1-\upsilon}z_{2,t}^*\right) - \frac{\upsilon}{(1-\upsilon)}z_{1,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C}g_t$$
$$= \frac{1}{(1-\upsilon)\sigma_C} y_t - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}c_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_t - \frac{\upsilon}{(1-\upsilon)}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_{2,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C}g_t$$

, which can be rewritten as:

$$\frac{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}c_t = \frac{1}{(1-\upsilon)\sigma_C}y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_t - \frac{\upsilon}{(1-\upsilon)}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_{2,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C}g_t$$

(4-13-30)'''

. Finally:

$$c_t = \frac{(1-\upsilon)^2}{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)}\left[\frac{1}{(1-\upsilon)\sigma_C}y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_t - \frac{\upsilon}{(1-\upsilon)}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2}z_{2,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C}g_t\right]$$
$$= \frac{(1-\upsilon)}{\left[(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)\right]\sigma_C}y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)}z_t - \frac{\upsilon(1-\upsilon)}{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)}z_{1,t}^*$$
$$- \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)}z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\left[(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)\right]\sigma_C}g_t$$

Plugging eq. (4-13-30)'' into the previous expression yields:

$$c_t = \frac{(1-\upsilon)(1-\alpha)}{\gamma_\upsilon\sigma_C}n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}z_{2,t}^*$$
$$- \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C}g_t + \frac{(1-\upsilon)}{\gamma_\upsilon\sigma_C}a_t \quad . \text{ (4-13-30)}$$

with $\gamma_\upsilon \equiv (1-\upsilon)^2 + \eta\upsilon(2-\upsilon)$.

Plugging Eq. (4-13-30) into Eq.(4-13-30)'' yields:

$$s_t = \frac{1}{1-\upsilon}\begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z^*_{1,t} \\ -\frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z^*_{2,t} - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix}$$

$$-\frac{1}{1-\upsilon} z_t + \frac{1}{1-\upsilon} z^*_{2,t} \quad . \text{(4-13-32)}$$

$$= \frac{1}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon}{\gamma_\upsilon} z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z^*_{2,t}$$

$$-\frac{\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{1}{\gamma_\upsilon \sigma_C} a_t$$

Plugging Eq.(4-13-32) into Eq.(4-8-3) yields:

$$x_{H,t} = -\upsilon \begin{bmatrix} \frac{1}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon}{\gamma_\upsilon} z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z^*_{2,t} \\ -\frac{\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{1}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix}$$

$$= -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z^*_{1,t} + \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z^*_{2,t} \quad . \text{(4-13-31)}$$

$$+\frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C} g_t - \frac{\upsilon}{\gamma_\upsilon \sigma_C} a_t$$

Let plug Eqs.(4-13-30) to (4-13-32) and (4-1-30)" into the second-order and cross terms in Eq.(4-13-35). Then we have:

$$x^2_{H,t} = \begin{Bmatrix} -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z^*_{1,t} + \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z^*_{2,t} \\ +\frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C} g_t - \frac{\upsilon}{\gamma_\upsilon \sigma_C} a_t \end{Bmatrix}^2$$

$$= \frac{\upsilon^2(1-\alpha)^2}{(\gamma_\upsilon \sigma_C)^2} n^2_t + \frac{2\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon](1-\alpha)}{(1-\upsilon)\gamma^2_\upsilon \sigma_C} n_t z_t - \frac{2\upsilon^3(1-\alpha)}{\gamma^2_\upsilon \sigma_C} n_t z^*_{1,t} \quad , \text{(4-13-36)}$$

$$-\frac{2\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon](1-\alpha)}{(1-\upsilon)\gamma^2_\upsilon \sigma_C} n_t z^*_{2,t} - \frac{2\upsilon^2(1-\alpha)\sigma_G}{(\gamma_\upsilon \sigma_C)^2} n_t g_t + \frac{2\upsilon^2(1-\alpha)}{(\gamma_\upsilon \sigma_C)^2} n_t a_t$$

$$+\text{s.o.t.i.p}$$

$$y^2_t = [n_t + a_t]^2$$
$$= n^2_t + 2(1-\alpha) n_t a_t + \text{s.o.t.i.p} \quad , \text{(4-13-37)}$$

$$
\begin{aligned}
c_t^2 &= \begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta \upsilon (2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon (1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta \upsilon (2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \\ -\frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix}^2 \\
&= \frac{(1-\upsilon)^2}{(\gamma_\upsilon \sigma_C)^2} n_t^2 + \frac{2\upsilon \eta (2-\upsilon)(1-\upsilon)}{\gamma_\upsilon^2 \sigma_C} n_t z_t \\
&\quad - \frac{2\upsilon (1-\upsilon)^2}{\gamma_\upsilon^2 \sigma_C} n_t z_{1,t}^* - \frac{2\upsilon \eta (2-\upsilon)(1-\upsilon)}{\gamma_\upsilon^2 \sigma_C} n_t z_{2,t}^* \\
&\quad - \frac{2\upsilon (1-\upsilon)^2 \sigma_G}{(\gamma_\upsilon \sigma_C)^2} n_t g_t + \frac{2(1-\upsilon)^2}{(\gamma_\upsilon \sigma_C)^2} n_t a_t + \text{s.o.t.i.p}
\end{aligned}
\quad , \text{(4-13-38)}
$$

with $\gamma_C \equiv (1-\upsilon)^2$ and $\gamma_D \equiv \eta(2-\upsilon)(1-\upsilon)$,

$$
\begin{aligned}
{s_t}^2 &= \begin{bmatrix} \frac{1}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta \upsilon (2-\upsilon) - \gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta \upsilon (2-\upsilon) - \gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* \\ -\frac{\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{1}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix}^2 \\
&= \frac{1}{(\gamma_\upsilon \sigma_C)^2} n_t^2 + \frac{2[\eta \upsilon (2-\upsilon) - \gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2 \sigma_C} n_t z_t - \frac{\upsilon}{\gamma_\upsilon^2 \sigma_C} n_t z_{1,t}^* \\
&\quad - \frac{2[\eta \upsilon (2-\upsilon) - \gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2 \sigma_C} n_t z_{2,t}^* - \frac{\sigma_G}{(\gamma_\upsilon \sigma_C)^2} n_t g_t + \frac{1}{(\gamma_\upsilon \sigma_C)^2} n_t a_t \\
&\quad + \text{s.o.t.i.p}
\end{aligned}
$$

$$c_t x_{H,t} = \begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t \\ -\frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \\ -\frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix} \begin{bmatrix} -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t \\ -\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z_{1,t}^* \\ +\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C} g_t \\ -\frac{\upsilon}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix}$$

$$= -\frac{\upsilon(1-\upsilon)}{(\gamma_\upsilon \sigma_C)^2} n_t^2 - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{\gamma_\upsilon^2 \sigma_C} n_t z_t + \frac{\upsilon^2(1-\upsilon)}{\gamma_\upsilon^2 \sigma_C} n_t z_{1,t}^*$$

$$+\frac{\upsilon(1-\upsilon)[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2 \sigma_C} n_t z_{2,t}^* + \frac{\upsilon(1-\upsilon)\sigma_G}{(\gamma_\upsilon \sigma_C)^2} n_t g_t$$

$$-\frac{\upsilon(1-\upsilon)}{(\gamma_\upsilon \sigma_C)^2} n_t a_t - \frac{\upsilon^2\eta(2-\upsilon)}{\gamma_\upsilon^2 \sigma_C} n_t z_t + \frac{\upsilon^2(1-\upsilon)}{\gamma_\upsilon^2 \sigma_C} n_t z_{1,t}^*$$

$$+\frac{\upsilon^2\eta(2-\upsilon)}{\gamma_\upsilon^2 \sigma_C} n_t z_{2,t}^* + \frac{\upsilon(1-\upsilon)\sigma_G}{(\gamma_\upsilon \sigma_C)^2} n_t g_t - \frac{\upsilon(1-\upsilon)}{(\gamma_\upsilon \sigma_C)^2} n_t a_t + \text{s.o.t.i.p}$$

$$= -\frac{\upsilon(1-\upsilon)}{(\gamma_\upsilon \sigma_C)^2} n_t^2 - \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]}{\gamma_\upsilon^2 \sigma_C} n_t z_t + \frac{\upsilon^2 2(1-\upsilon)}{\gamma_\upsilon^2 \sigma_C} n_t z_{1,t}^* +$$

$$\frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]}{\gamma_\upsilon^2 \sigma_C} n_t z_{2,t}^* + \frac{\upsilon 2(1-\upsilon)\sigma_G}{(\gamma_\upsilon \sigma_C)^2} n_t g_t - \frac{\upsilon 2(1-\upsilon)}{(\gamma_\upsilon \sigma_C)^2} n_t a_t + \text{s.o.t.i.p} \quad \text{, (4-13-39)}$$

$$x_{H,t} y_t = \begin{Bmatrix} -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z_{1,t}^* \\ +\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C} g_t - \frac{\upsilon}{\gamma_\upsilon \sigma_C} a_t \end{Bmatrix} [n_t + a_t]$$

$$= -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t^2 - \frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t a_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t n_t + \frac{\upsilon^2}{\gamma_\upsilon} n_t z_{1,t}^* \quad \text{, (4-13-40)}$$

$$+\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} n_t z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C} n_t g_t - \frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t a_t + \text{s.o.t.i.p}$$

$$c_t y_t = \begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* \\ -\frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix} [n_t + a_t]$$

$$= \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t^2 + \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t a_t + \frac{\upsilon\eta(2-\upsilon)}{\gamma_\upsilon} n_t z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} n_t z_{1,t}^* \quad , \text{(4-13-41)}$$

$$\frac{\upsilon\eta(2-\upsilon)}{\gamma_\upsilon} n_t z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} n_t g_t + \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} n_t a_t + \text{s.o.t.i.p}$$

$$c_t n_t = \begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \\ -\frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix} n_t$$

$$= \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t^2 + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} n_t z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} n_t z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} n_t z_{2,t}^* \text{ , (4-13-42)}$$

$$-\frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} n_t g_t + \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} n_t a_t$$

$$n_t x_{H,t} = n_t \begin{Bmatrix} -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z_{1,t}^* \\ +\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C} g_t - \frac{\upsilon}{\gamma_\upsilon \sigma_C} a_t \end{Bmatrix}$$

$$= -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t^2 - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} n_t z_t + \frac{\upsilon^2}{\gamma_\upsilon} n_t z_{1,t}^* \quad , \text{(4-13-43)}$$

$$+\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} n_t z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C} n_t g_t - \frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t a_t$$

$$x_{H,t} g_t = \begin{Bmatrix} -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z_{1,t}^* \\ +\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C} g_t - \frac{\upsilon}{\gamma_\upsilon \sigma_C} a_t \end{Bmatrix} g_t \quad , \text{(4-13-49)}$$

$$= -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t g_t + \text{s.o.t.i.p}$$

$$c_t g_t = \begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t \\ + \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix} g_t \qquad \text{, (4-13-50)}$$

$$= \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t g_t + \text{s.o.t.i.p}$$

$$c_t a_t = \begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t \\ + \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix} a_t \qquad \text{, (4-13-51)}$$

$$= \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t a_t + \text{t.i.p}$$

$$y_t a_t = [n_t + a_t] a_t \qquad \text{, (4-13-49)'}$$

$$= n_t a_t + \text{s.o.t.i.p}$$

$$x_t a_t = \begin{Bmatrix} -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t - \frac{\upsilon[\eta\upsilon(2-\upsilon) - \gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z_{1,t}^* \\ + \frac{\upsilon[\eta\upsilon(2-\upsilon) - \gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C} g_t - \frac{\upsilon}{\gamma_\upsilon \sigma_C} a_t \end{Bmatrix} a_t \qquad \text{, (4-13-52)}$$

$$= -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t a_t + \text{s.o.t.i.p}$$

$$c_t \hat{\zeta}_t = \begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \\ - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix} \hat{\zeta}_t \qquad \text{, (4-13-53)}$$

$$= \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t \hat{\zeta}_t + \text{s.o.t.i.p}$$

$$c_t z_t = \begin{bmatrix} \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \\ - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t + \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} a_t \end{bmatrix} z_t \qquad \text{, (4-13-53b)}$$

$$= \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t z_t + \text{s.o.t.i.p}$$

$$x_{H,t}z_t = \left\{ \begin{array}{l} -\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z_t + \frac{\upsilon^2}{\gamma_\upsilon} z^*_{1,t} \\ + \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon} z^*_{2,t} + \frac{\upsilon\sigma_G}{\gamma_\upsilon\sigma_C} g_t - \frac{\upsilon}{\gamma_\upsilon\sigma_C} a_t \end{array} \right\} z_t \\ = -\frac{\upsilon}{\gamma_\upsilon\sigma_C} n_t z_t + \text{s.o.t.i.p}, \quad \text{(4-13-53c)}$$

$$y_t z_t = [n_t + a_t] z_t \\ = n_t z_t + \text{s.o.t.i.p}, \quad \text{(4-13-53b)}$$

$$s_t z^*_{1,t} = \left[\begin{array}{l} \frac{1}{\gamma_\upsilon\sigma_C} n_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon}{\gamma_\upsilon} z^*_{1,t} - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z^*_{2,t} \\ -\frac{\sigma_G}{\gamma_\upsilon\sigma_C} g_t + \frac{1}{\gamma_\upsilon\sigma_C} a_t \end{array} \right] z^*_{1,t} \\ = \frac{1}{\gamma_\upsilon\sigma_C} n_t z^*_{1,t} + \text{s.o.t.i.p}, \quad \text{(4-13-53-e)}$$

where we use Eqs.(4-11-6) and (4-11-7) to derive Eq.(4-13-36).

Now we calculate coefficients of in Eq.(4-13-35). These are given by:

$$\left[\frac{1}{2}\Theta_1 - \frac{1}{2(1-\tau)}\Theta_2 + \frac{\eta^2(1-\upsilon)\sigma_C}{2}\Theta_3\right] x_t^2 = \frac{(1-\tau)\Theta_1 - \Theta_2 + (1-\tau)\eta^2(1-\upsilon)\sigma_C\Theta_3}{2(1-\tau)} x_t^2 \\ = \frac{(1-\tau)[\Theta_1 + \eta^2(1-\upsilon)\sigma_C\Theta_3] - \Theta_2}{2(1-\tau)} x_t^2,$$

$$\left[\frac{\Theta_1\tau}{2(1-\beta)\sigma_B} + \frac{\Theta_2[\varphi(1-\tau)-\tau]}{(1-\tau)2}\right] y_t^2 = \frac{(1-\tau)\Theta_1 + \Theta_2(1-\beta)\sigma_B[\varphi(1-\tau)-\tau]}{2(1-\beta)(1-\tau)\sigma_B} y_t^2$$

$$\left[\frac{\Theta_1}{2} + \frac{\tau-(1-\tau)}{2(1-\tau)}\Theta_2 + \frac{\Theta_3(1-\upsilon)\sigma_C}{2}\right] c_t^2 = \frac{(1-\tau)\Theta_1 - \tau\Theta_2 + (1-\tau)(1-\upsilon)\sigma_C\Theta_3}{2(1-\tau)} c_t^2 \\ = \frac{(1-\tau)\Theta_1 + [\tau-(1-\tau)]\Theta_2 + (1-\tau)(1-\upsilon)\sigma_C\Theta_3}{2(1-\tau)} c_t^2 \\ = \frac{(1-\tau)[\Theta_1 + \sigma_C\Theta_3(1-\upsilon)] - (1-2\tau)\Theta_2}{2(1-\tau)} c_t^2$$

$$\left[\frac{\Theta_3\upsilon\eta^2\sigma_C}{2} + \frac{\Theta_4\upsilon[\eta(1-\upsilon)-1]}{2}\right] s_t^2 = \frac{\upsilon\{\Theta_3\eta^2\sigma_C + \Theta_4[\eta(1-\upsilon)-1]\}}{2} s_t^2$$
, and we define as

follows:

$$x_{H,t}^2: \ \Gamma_{11} \equiv \frac{(1-\tau)\left[\Theta_1 + \eta^2 (1-\upsilon)\sigma_C \Theta_3\right] - \Theta_2}{2(1-\tau)}, \text{(4-13-54)}$$

$$y_t^2: \ \Gamma_{12} \equiv \frac{(1-\tau)\Theta_1 + \Theta_2 (1-\beta)\sigma_B \left[\varphi(1-\tau) - \tau\right]}{2(1-\beta)(1-\tau)\sigma_B}, \text{(4-13-55)}$$

$$c_t^2: \ \Gamma_{13} \equiv \frac{(1-\tau)\left[\Theta_1 + \sigma_C \Theta_3 (1-\upsilon)\right] - (1-2\tau)\Theta_2}{2(1-\tau)}, \text{(4-13-56)}$$

$$s_t^2: \ \Gamma_{14} \equiv \frac{\Theta_3 \eta^2 \sigma_C + \Theta_4 \left[\eta(1-\upsilon) - 1\right]}{2}, \text{(4-13-57)}$$

$$c_t x_{H,t}: \ \Gamma_{15} \equiv \Theta_1 - \Theta_2 + \Theta_3 \eta (1-\upsilon)\sigma_C, \text{(4-13-58)}.$$

By Plugging Eqs.(4-13-36) to (4-13-39) and (4-13-54) to (4-14-58) into lines 3 to 7 in Eq. (4-14-35), we have:

$$
\left[\frac{\Theta_1}{2}-\frac{\Theta_2}{2(1-\tau)}+\frac{\Theta_3\eta^2(1-\upsilon)\sigma_C}{2}\right]x_{H,t}^2+\left[\frac{\Theta_1\tau}{2(1-\beta)\sigma_B}+\frac{\Theta_2[\varphi(1-\tau)-\tau]}{(1-\tau)2}\right]y_t^2
$$

$$
+\left[\frac{\Theta_1}{2}+\frac{\tau-(1-\tau)\Theta_2}{2(1-\tau)}+\frac{\Theta_3(1-\upsilon)\sigma_C}{2}\right]c_t^2+\left[\frac{\Theta_3\upsilon\eta^2\sigma_C}{2}+\frac{\Theta_4\upsilon[\eta(1-\upsilon)-1]}{2}\right]s_t^2
$$

$$
-[\Theta_1-\Theta_2+\Theta_3\eta(1-\upsilon)\sigma_C]c_t x_{H,t}
$$

$$
=\Gamma_{11}\left\{\begin{array}{l}\frac{\upsilon^2}{(\gamma_\upsilon\sigma_C)^2}n_t^2+\frac{2\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_t z_t-\frac{2\upsilon^3}{\gamma_\upsilon^2\sigma_C}n_t z_{1,t}^*\\ -\frac{2\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_t z_{2,t}^*-\frac{2\upsilon^2\sigma_G}{(\gamma_\upsilon\sigma_C)^2}n_t g_t+\frac{2\upsilon^2}{(\gamma_\upsilon\sigma_C)^2}n_t a_t\end{array}\right\}
$$

$$
+\Gamma_{12}\left(n_t^2+2n_t a_{H,t}\right)
$$

$$
+\Gamma_{13}\left[\begin{array}{l}\frac{(1-\upsilon)^2}{(\gamma_\upsilon\sigma_C)^2}n_t^2+\frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)}{\gamma_\upsilon^2\sigma_C}n_t z_t-\frac{2\upsilon(1-\upsilon)^2}{\gamma_\upsilon^2\sigma_C}n_t z_{1,t}^*-\frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)}{\gamma_\upsilon^2\sigma_C}n_t z_{2,t}^*\\ -\frac{2\upsilon(1-\upsilon)^2\sigma_G}{(\gamma_\upsilon\sigma_C)^2}n_t g_t+\frac{2(1-\upsilon)^2}{(\gamma_\upsilon\sigma_C)^2}n_t a_t\end{array}\right]
$$

$$
+\Gamma_{14}\left\{\begin{array}{l}\frac{1}{(\gamma_\upsilon\sigma_C)^2}n_t^2+\frac{2(1-\alpha)[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_t z_t-\frac{\upsilon}{\gamma_\upsilon^2\sigma_C}n_t z_{1,t}^*\\ -\frac{2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_t z_{2,t}^*-\frac{\sigma_G}{(\gamma_\upsilon\sigma_C)^2}n_t g_t+\frac{1}{(\gamma_\upsilon\sigma_C)^2}n_t a_t\end{array}\right\}
$$

$$
-\Gamma_{15}\left\{\begin{array}{l}-\frac{\upsilon(1-\upsilon)}{(\gamma_\upsilon\sigma_C)^2}n_t^2-\frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]}{\gamma_\upsilon^2\sigma_C}n_t z_t+\frac{\upsilon^2 2(1-\upsilon)}{\gamma_\upsilon^2\sigma_C}n_t z_{1,t}^*+\frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]}{\gamma_\upsilon^2\sigma_C}n_t z_{2,t}^*\\ +\frac{\upsilon 2(1-\upsilon)\sigma_G}{(\gamma_\upsilon\sigma_C)^2}n_t g_t-\frac{\upsilon 2(1-\upsilon)}{(\gamma_\upsilon\sigma_C)^2}n_t a_t\end{array}\right\}
$$

$$
\begin{aligned}
&= \frac{\upsilon^2\Gamma_{11}}{(\gamma_\upsilon\sigma_C)^2}n_t^2 + \frac{2\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{11}}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_tz_t - \frac{2\upsilon^3\Gamma_{11}}{\gamma_\upsilon^2\sigma_C}n_tz_{1,t}^* - \frac{2\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{11}}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_tz_{2,t}^* \\
&- \frac{2\upsilon^2\sigma_G\Gamma_{11}}{(\gamma_\upsilon\sigma_C)^2}n_tg_t + \frac{2\upsilon^2\Gamma_{11}}{(\gamma_\upsilon\sigma_C)^2}n_ta_t + \Gamma_{12}n_t^2 + 2\Gamma_{12}n_ta_t + \frac{(1-\upsilon)^2\Gamma_{13}}{(\gamma_\upsilon\sigma_C)^2}n_t^2 \\
&+ \frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}}{\gamma_\upsilon^2\sigma_C}n_tz_t - \frac{2\upsilon(1-\upsilon)^2\Gamma_{13}}{\gamma_\upsilon^2\sigma_C}n_tz_{1,t}^* - \frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}}{\gamma_\upsilon^2\sigma_C}n_tz_{2,t}^* \\
&- \frac{2\upsilon(1-\upsilon)^2\sigma_G\Gamma_{13}}{(\gamma_\upsilon\sigma_C)^2}n_tg_t + \frac{2(1-\upsilon)^2\Gamma_{13}}{(\gamma_\upsilon\sigma_C)^2}n_ta_t + \frac{\Gamma_{14}}{(\gamma_\upsilon\sigma_C)^2}n_t^2 + \frac{2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{14}}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_tz_t \\
&- \frac{\upsilon\Gamma_{14}}{\gamma_\upsilon^2\sigma_C}n_tz_{1,t}^* - \frac{2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{14}}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_tz_{2,t}^* - \frac{\sigma_G\Gamma_{14}}{(\gamma_\upsilon\sigma_C)^2}n_tg_t + \frac{\Gamma_{14}}{(\gamma_\upsilon\sigma_C)^2}n_ta_t + \frac{\upsilon(1-\upsilon)\Gamma_{15}}{(\gamma_\upsilon\sigma_C)^2}n_t^2 \\
&+ \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]\Gamma_{15}}{\gamma_\upsilon^2\sigma_C}n_tz_t - \frac{\upsilon^2 2(1-\upsilon)\Gamma_{15}}{\gamma_\upsilon^2\sigma_C}n_tz_{1,t}^* - \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]\Gamma_{15}}{\gamma_\upsilon^2\sigma_C}n_tz_{2,t}^* \\
&- \frac{\upsilon 2(1-\upsilon)\sigma_G\Gamma_{15}}{(\gamma_\upsilon\sigma_C)^2}n_tg_t + \frac{\upsilon 2(1-\upsilon)\Gamma_{15}}{(\gamma_\upsilon\sigma_C)^2}n_ta_t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\upsilon^2\Gamma_{11}}{(\gamma_\upsilon\sigma_C)^2}n_t^2 + \Gamma_{12}n_t^2 + \frac{(1-\upsilon)^2\Gamma_{13}}{(\gamma_\upsilon\sigma_C)^2}n_t^2 + \frac{\Gamma_{14}}{(\gamma_\upsilon\sigma_C)^2}n_t^2 + \frac{\upsilon(1-\upsilon)\Gamma_{15}}{(\gamma_\upsilon\sigma_C)^2}n_t^2 \\
&+ \frac{2\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{11}}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_tz_t + \frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}}{\gamma_\upsilon^2\sigma_C}n_tz_t \\
&+ \frac{2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{14}}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_tz_t + \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]\Gamma_{15}}{\gamma_\upsilon^2\sigma_C}n_tz_t - \frac{2\upsilon^3\Gamma_{11}}{\gamma_\upsilon^2\sigma_C}n_tz_{1,t}^* \\
&- \frac{2\upsilon(1-\upsilon)^2\Gamma_{13}}{\gamma_\upsilon^2\sigma_C}n_tz_{1,t}^* - \frac{\upsilon\Gamma_{14}}{\gamma_\upsilon^2\sigma_C}n_tz_{1,t}^* - \frac{\upsilon^2 2(1-\upsilon)\Gamma_{15}}{\gamma_\upsilon^2\sigma_C}n_tz_{1,t}^* - \frac{2\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{11}}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_tz_{2,t}^* \\
&- \frac{2\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}}{\gamma_\upsilon^2\sigma_C}n_tz_{2,t}^* - \frac{2[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Gamma_{14}}{(1-\upsilon)\gamma_\upsilon^2\sigma_C}n_tz_{2,t}^* - \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_\upsilon]\Gamma_{15}}{\gamma_\upsilon^2\sigma_C}n_tz_{2,t}^* \\
&- \frac{2\upsilon^2\sigma_G\Gamma_{11}}{(\gamma_\upsilon\sigma_C)^2}n_tg_t - \frac{2\upsilon(1-\upsilon)^2\sigma_G\Gamma_{13}}{(\gamma_\upsilon\sigma_C)^2}n_tg_t - \frac{\sigma_G\Gamma_{14}}{(\gamma_\upsilon\sigma_C)^2}n_tg_t - \frac{\upsilon 2(1-\upsilon)\sigma_G\Gamma_{15}}{(\gamma_\upsilon\sigma_C)^2}n_tg_t \\
&+ \frac{2\upsilon^2\Gamma_{11}}{(\gamma_\upsilon\sigma_C)^2}n_ta_t + 2\Gamma_{12}n_ta_t + \frac{2(1-\upsilon)^2\Gamma_{13}}{(\gamma_\upsilon\sigma_C)^2}n_ta_t + \frac{\Gamma_{14}}{(\gamma_\upsilon\sigma_C)^2}n_ta_t + \frac{\upsilon 2(1-\upsilon)\Gamma_{15}}{(\gamma_\upsilon\sigma_C)^2}n_ta_t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(\gamma_v \sigma_C)^2}\left[\upsilon^2\Gamma_{11} + (\gamma_v\sigma_C)^2\Gamma_{12} + (1-\upsilon)^2\Gamma_{13} + \Gamma_{14} + \upsilon(1-\upsilon)\Gamma_{15}\right] n_t^2 \\
&- \frac{2}{\gamma_v^2\sigma_C}\left\{\begin{array}{l} -\frac{\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{11}}{1-\upsilon} n_t z_t - \upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13} n_t z_t \\ -\frac{[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{14}}{1-\upsilon} n_t z_t - \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_v]\Gamma_{15}}{2} \end{array}\right\} n_t z_t \\
&- \frac{2\upsilon}{\gamma_v^2\sigma_C}\left[\upsilon^2\Gamma_{11} + (1-\upsilon)^2\Gamma_{13} + \frac{\Gamma_{14}}{2} + \upsilon(1-\upsilon)\Gamma_{15}\right] n_t z_{1,t}^* \\
&- \frac{2}{\gamma_v^2\sigma_C}\left[\begin{array}{l} \frac{\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{11}}{(1-\upsilon)} + \upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13} \\ + \frac{[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{14}}{(1-\upsilon)} + \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_v]\Gamma_{15}}{2} \end{array}\right] n_t z_{2,t}^* \\
&- \frac{2\sigma_G}{(\gamma_v\sigma_C)^2}\left[\upsilon^2\Gamma_{11} + \upsilon(1-\upsilon)^2\Gamma_{13} + \frac{\Gamma_{14}}{2} + \upsilon(1-\upsilon)\Gamma_{15}\right] n_t g_t \\
&- \frac{2}{(\gamma_v\sigma_C)^2}\left[-\upsilon^2\Gamma_{11} - (\gamma_v\sigma_C)^2\Gamma_{12} - (1-\upsilon)^2\Gamma_{13} - \frac{\Gamma_{14}}{2} - \upsilon(1-\upsilon)\Gamma_{15}\right] n_t a_t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(\gamma_v \sigma_C)^2}\left\{\upsilon^2\Gamma_{11} + (1-\upsilon)^2\Gamma_{13} + \Gamma_{14} + (\gamma_v\sigma_C)^2\Gamma_{12} + \upsilon(1-\upsilon)\Gamma_{15}\right\} n_t^2 \\
&- \frac{2}{\gamma_v^2\sigma_C}\left\{\begin{array}{l} -\frac{\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{11}}{1-\upsilon} n_t z_t - \upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13} n_t z_t \\ -\frac{[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{14}}{1-\upsilon} n_t z_t - \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_v]\Gamma_{15}}{2} \end{array}\right\} n_t z_t \\
&- \frac{2\upsilon}{\gamma_v^2\sigma_C}\left[\upsilon^2\Gamma_{11} + (1-\upsilon)^2\Gamma_{13} + \frac{\Gamma_{14}}{2} + \upsilon(1-\upsilon)\Gamma_{15}\right] n_t z_{1,t}^* \\
&- \frac{2}{\gamma_v^2\sigma_C}\left[\begin{array}{l} \frac{\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{11}}{(1-\upsilon)} + \upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13} \\ + \frac{[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{14}}{(1-\upsilon)} + \frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_v]\Gamma_{15}}{2} \end{array}\right] n_t z_{2,t}^* \\
&- \frac{2\sigma_G}{(\gamma_v\sigma_C)^2}\left[\upsilon^2\Gamma_{11} + \upsilon(1-\upsilon)^2\Gamma_{13} + \frac{\Gamma_{14}}{2} + \upsilon(1-\upsilon)\Gamma_{15}\right] n_t g_t \\
&- \frac{2}{(\gamma_v\sigma_C)^2}\left[-\upsilon^2\Gamma_{11} - (\gamma_v\sigma_C)^2\Gamma_{12} - (1-\upsilon)^2\Gamma_{13} - \frac{\Gamma_{14}}{2} - \upsilon(1-\upsilon)\Gamma_{15}\right] n_t a_t
\end{aligned}
$$

$$=\frac{1-\alpha}{(\gamma_v\sigma_C)^2}\left\{(1-\alpha)\left[\upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\Gamma_{14}\right]+\frac{(\gamma_v\sigma_C)^2}{1-\alpha}\Gamma_{12}+\frac{\upsilon(1-\upsilon)}{1-\alpha}\Gamma_{15}\right\}n_t^2$$

$$-\frac{2(1-\alpha)}{\gamma_v^2\sigma_C}\left\{\begin{array}{l}-\frac{\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{11}}{1-\upsilon}n_tz_t-\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}n_tz_t\\-\frac{[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{14}}{1-\upsilon}n_tz_t-\frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_v]\Gamma_{15}}{2}\end{array}\right\}n_tz_t$$

$$-\frac{2\upsilon(1-\alpha)}{\gamma_v^2\sigma_C}\left[\begin{array}{l}\upsilon^2\Gamma_{11}+(1-\upsilon)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}\\+\upsilon(1-\upsilon)\Gamma_{15}\end{array}\right]n_tz_{1,t}^*$$

$$-\frac{2(1-\alpha)}{\gamma_v^2\sigma_C}\left[\begin{array}{l}\frac{\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{11}}{(1-\upsilon)}+\upsilon\eta(2-\upsilon)(1-\upsilon)\Gamma_{13}\\+\frac{[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{14}}{(1-\upsilon)}+\frac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_v]\Gamma_{15}}{2}\end{array}\right]n_tz_{2,t}^*$$

$$-\frac{2(1-\alpha)\sigma_G}{(\gamma_v\sigma_C)^2}\left[\begin{array}{l}\upsilon^2\Gamma_{11}+\upsilon(1-\upsilon)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}\\+\upsilon(1-\upsilon)\Gamma_{15}\end{array}\right]n_tg_t$$

$$-\frac{2(1-\alpha)}{(\gamma_v\sigma_C)^2}\left[\begin{array}{l}-\upsilon^2\Gamma_{11}-(\gamma_v\sigma_C)^2\Gamma_{12}-(1-\upsilon)^2\Gamma_{13}\\-\frac{\Gamma_{14}}{2}-\upsilon(1-\upsilon)\Gamma_{15}\end{array}\right]n_ta_t$$

(4-13-35a)

By Plugging Eqs. (4-13-40) to (4-13-53e) into lines 8 to 12 in Eq. (4-14-35), we have:

$$\frac{\Theta_1\tau}{(1-\beta)\sigma_B}x_{H,t}y_t-\frac{\Theta_1\tau}{(1-\beta)\sigma_B}c_ty_t+\Theta_2c_tn_t-\Theta_2n_tx_{H,t}-\frac{\Theta_1\sigma_G}{(1-\beta)\sigma_B}x_{H,t}g_t$$

$$+\frac{\Theta_1\sigma_G}{(1-\beta)\sigma_B}c_tg_t+\Theta_2c_ta_t-\varphi\Theta_2y_ta_t-\Theta_2x_{H,t}a_t$$

$$+\frac{\Theta_1}{(1-\beta)\sigma_B}c_t\hat{\zeta}_t-\Theta_1c_tz_t+\Theta_1x_{H,t}z_t+\frac{\Theta_1\tau}{(1-\beta)\sigma_B}y_tz_t+\Theta_3\upsilon\eta\sigma_Cs_tz_{1,t}^*$$

$$
\begin{aligned}
&= \frac{\Theta_1\tau}{(1-\beta)\sigma_B}\begin{Bmatrix} -\frac{\upsilon}{\gamma_\upsilon\sigma_C}n_t^2 - \frac{\upsilon}{\gamma_\upsilon\sigma_C}n_t a_t - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon}z_t n_t + \frac{\upsilon^2}{\gamma_\upsilon}n_t z_{1,t}^* \\ + \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon}n_t z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon\sigma_C}n_t g_t - \frac{\upsilon}{\gamma_\upsilon\sigma_C}n_t a_t \end{Bmatrix} \\
&- \frac{\Theta_1\tau}{(1-\beta)\sigma_B}\begin{Bmatrix} \frac{1-\upsilon}{\gamma_\upsilon\sigma_C}n_t^2 + \frac{1-\upsilon}{\gamma_\upsilon\sigma_C}n_t a_t + \frac{\upsilon\eta(2-\upsilon)}{\gamma_\upsilon}n_t z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}n_t z_{1,t}^* \\ - \frac{\upsilon\eta(2-\upsilon)}{\gamma_\upsilon}n_t z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C}n_t g_t + \frac{1-\upsilon}{\gamma_\upsilon\sigma_C}n_t a_t \end{Bmatrix} \\
&+ \Theta_2\begin{Bmatrix} \frac{1-\upsilon}{\gamma_\upsilon\sigma_C}n_t^2 + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}n_t z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}n_t z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon}n_t z_{2,t}^* \\ - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C}n_t g_t + \frac{1-\upsilon}{\gamma_\upsilon\sigma_C}n_t a_t \end{Bmatrix} \\
&- \Theta_2\begin{Bmatrix} -\frac{\upsilon}{\gamma_\upsilon\sigma_C}n_t^2 - \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon}n_t z_t + \frac{\upsilon^2}{\gamma_\upsilon}n_t z_{1,t}^* \\ + \frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]}{(1-\upsilon)\gamma_\upsilon}n_t z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon\sigma_C}n_t g_t - \frac{\upsilon}{\gamma_\upsilon\sigma_C}n_t a_t \end{Bmatrix} \\
&- \frac{\Theta_1\sigma_G}{(1-\beta)\sigma_B}\left[-\frac{\upsilon}{\gamma_\upsilon\sigma_C}n_t g_t\right] + \frac{\Theta_1\sigma_G}{(1-\beta)\sigma_B}\left[\frac{(1-\upsilon)}{\gamma_\upsilon\sigma_C}n_t g_t\right] + \frac{\Theta_2(1-\upsilon)}{\gamma_\upsilon\sigma_C}n_t a_t - \varphi\Theta_2 n_t a_t \\
&- \Theta_2\left[-\frac{\upsilon}{\gamma_\upsilon\sigma_C}n_t a_t\right] + \frac{\Theta_1}{(1-\beta)\sigma_B}\frac{(1-\upsilon)}{\gamma_\upsilon\sigma_C}n_t\hat{\zeta}_t - \frac{\Theta_1(1-\upsilon)}{\gamma_\upsilon\sigma_C}n_t z_t + \Theta_1\left[-\frac{\upsilon}{\gamma_\upsilon\sigma_C}n_t z_t\right] \\
&+ \frac{\Theta_1\tau}{(1-\beta)\sigma_B}n_t z_t + \frac{\Theta_3\upsilon\eta\sigma_C}{\gamma_\upsilon\sigma_C}n_t z_{1,t}^*
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\upsilon}{\gamma_\upsilon \sigma_C}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t^2 - \frac{\upsilon}{\gamma_\upsilon \sigma_C}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t a_t - \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} z_t n_t \\
&+ \frac{\upsilon^2}{\gamma_\upsilon}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t z_{1,t}^* + \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t z_{2,t}^* + \frac{\upsilon\sigma_G}{\gamma_\upsilon \sigma_C}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t g_t \\
&- \frac{\upsilon}{\gamma_\upsilon \sigma_C}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t a_t - \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t^2 - \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t a_t \\
&- \frac{\upsilon\eta(2-\upsilon)}{\gamma_\upsilon}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t z_t + \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t z_{1,t}^* + \frac{\upsilon\eta(2-\upsilon)}{\gamma_\upsilon}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t z_{2,t}^* \\
&+ \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t g_t - \frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C}\frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t a_t + \frac{(1-\upsilon)\Theta_2}{\gamma_\upsilon \sigma_C} n_t^2 \\
&+ \frac{\eta\upsilon(2-\upsilon)\Theta_2}{\gamma_\upsilon} n_t z_t - \frac{\upsilon(1-\upsilon)\Theta_2}{\gamma_\upsilon} n_t z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)\Theta_2}{\gamma_\upsilon} n_t z_{2,t}^* - \frac{(1-\upsilon)\sigma_G\Theta_2}{\gamma_\upsilon \sigma_C} n_t g_t \\
&+ \frac{(1-\upsilon)\Theta_2}{\gamma_\upsilon \sigma_C} n_t a_t + \frac{\upsilon\Theta_2}{\gamma_\upsilon \sigma_C} n_t^2 + \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Theta_2}{(1-\upsilon)\gamma_\upsilon} n_t z_t - \frac{\upsilon^2\Theta_2}{\gamma_\upsilon} n_t z_{1,t}^* \\
&- \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]\Theta_2}{(1-\upsilon)\gamma_\upsilon} n_t z_{2,t}^* - \frac{\upsilon\sigma_G\Theta_2}{\gamma_\upsilon \sigma_C} n_t g_t + \frac{\upsilon\Theta_2}{\gamma_\upsilon \sigma_C} n_t a_t + \frac{\Theta_1\sigma_G}{(1-\beta)\sigma_B}\frac{\upsilon}{\gamma_\upsilon \sigma_C} n_t g_t \\
&+ \frac{\Theta_1\sigma_G}{(1-\beta)\sigma_B}\frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} n_t g_t + \frac{\Theta_2(1-\upsilon)}{\gamma_\upsilon \sigma_C} n_t a_t - \varphi\Theta_2 n_t a_t + \frac{\upsilon\Theta_2}{\gamma_\upsilon \sigma_C} n_t a_t \\
&+ \frac{\Theta_1}{(1-\beta)\sigma_B}\frac{(1-\upsilon)}{\gamma_\upsilon \sigma_C} n_t \hat{\zeta}_t - \frac{\Theta_1(1-\upsilon)}{\gamma_\upsilon \sigma_C} n_t z_t - \frac{\upsilon\Theta_1}{\gamma_\upsilon \sigma_C} n_t z_t + \frac{\Theta_1 \tau}{(1-\beta)\sigma_B} n_t z_t + \frac{\Theta_3\upsilon\eta\sigma_C}{\gamma_\upsilon \sigma_C} n_t z_{1,t}^*
\end{aligned}
$$

$$=\frac{1}{(\gamma_\upsilon\sigma_C)^2}\left[-\frac{\Theta_1\tau\gamma_\upsilon\sigma_C}{(1-\beta)\sigma_B}+\Theta_2\gamma_\upsilon\sigma_C\right]n_t^2$$

$$-\frac{2}{\gamma_\upsilon}\left[\begin{array}{l}\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\tau\Theta_1}{2(1-\upsilon)(1-\beta)\sigma_B}-\frac{\gamma_\upsilon\tau\Theta_1}{2(1-\beta)\sigma_B}+\frac{\upsilon\eta(2-\upsilon)\tau\Theta_1}{2(1-\beta)\sigma_B}-\frac{\eta\upsilon(2-\upsilon)\Theta_2}{2}\\+\frac{\upsilon(1-\alpha)\Theta_1}{2\sigma_C}-\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{2(1-\upsilon)}+\frac{(1-\upsilon)\Theta_1}{2\sigma_C}\end{array}\right]n_tz_t$$

$$-\frac{2\upsilon}{\gamma_\upsilon}\left[-\frac{\tau\Theta_1}{(1-\beta)\sigma_B2}+\frac{\Theta_2}{2}-\frac{\eta\Theta_3}{2}\right]n_tz_{1,t}^*$$

$$-\frac{2\upsilon}{\gamma_\upsilon}\left[-\frac{[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\tau\Theta_1}{(1-\upsilon)2(1-\beta)\sigma_B}-\frac{\eta(2-\upsilon)\tau\Theta_1}{2(1-\beta)\sigma_B}+\frac{\eta(2-\upsilon)\Theta_2}{2}+\frac{[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{(1-\upsilon)2}\right]n_tz_{2,t}^*$$

$$-\frac{2\sigma_G}{\gamma_\upsilon\sigma_C}\left[-\frac{\Theta_1\tau}{2(1-\beta)\sigma_B}+\frac{\Theta_2}{2}-\frac{\upsilon\Theta_1}{2(1-\beta)\sigma_B}-\frac{(1-\upsilon)\Theta_1}{2(1-\beta)\sigma_B}\right]n_tg_t$$

$$-\frac{2}{\gamma_\upsilon\sigma_C}\left[\begin{array}{l}\frac{\upsilon\Theta_1\tau}{2(1-\beta)\sigma_B}+\frac{(1-\upsilon)\Theta_1\tau}{2(1-\beta)\sigma_B}+\frac{\upsilon\Theta_1\tau}{2(1-\beta)\sigma_B}+\frac{(1-\upsilon)\Theta_1\tau}{2(1-\beta)\sigma_B}\\-\frac{(1-\upsilon)\Theta_2}{2}-\frac{\upsilon\Theta_2}{2}-\frac{\Theta_2(1-\upsilon)}{2}+\frac{\Theta_2\gamma_\upsilon\sigma_C(1+\varphi)}{2}-\frac{\upsilon\Theta_2}{2}\end{array}\right]n_ta_t$$

$$+\frac{(1-\upsilon)\Theta_1}{(1-\beta)\sigma_B\gamma_\upsilon\sigma_C}n_t\hat{\zeta}_t$$

$$=\frac{1}{\gamma_\upsilon\sigma_C}\left[-\frac{\Theta_1\tau}{(1-\beta)\sigma_B}+\Theta_2\right]n_t^2$$

$$-\frac{2}{\gamma_\upsilon}\left[\begin{array}{l}\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\tau\Theta_1}{2(1-\upsilon)(1-\beta)\sigma_B}+\frac{\upsilon\eta(2-\upsilon)\tau\Theta_1}{2(1-\beta)\sigma_B}-\frac{\gamma_\upsilon\tau\Theta_1}{2(1-\beta)\sigma_B}-\frac{\eta\upsilon(2-\upsilon)\Theta_2}{2}+\frac{\Theta_1}{2\sigma_C}\\-\frac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{2(1-\upsilon)}\end{array}\right]n_tz_t$$

$$-\frac{2\upsilon}{\gamma_\upsilon^2\sigma_C}\left[-\frac{\gamma_\upsilon\sigma_C\tau\Theta_1}{(1-\beta)\sigma_B2}+\frac{\gamma_\upsilon\sigma_C\Theta_2}{2(1-\alpha)}-\frac{\eta\gamma_\upsilon\sigma_C\Theta_3}{2}\right]n_tz_{1,t}^*$$

$$-\frac{2\upsilon}{\gamma_\upsilon}\left[-\frac{[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\tau\Theta_1}{(1-\upsilon)2(1-\beta)\sigma_B}-\frac{\eta(2-\upsilon)\tau\Theta_1}{2(1-\beta)\sigma_B}+\frac{\eta(2-\upsilon)\Theta_2}{2}+\frac{[\eta\upsilon(2-\upsilon)-\gamma_\upsilon]\Theta_2}{(1-\upsilon)2}\right]n_tz_{2,t}^*$$

$$-\frac{2\sigma_G}{\gamma_\upsilon\sigma_C}\left[-\frac{\Theta_1\tau}{2(1-\beta)\sigma_B}+\frac{\Theta_2}{2}-\frac{\Theta_1}{2(1-\beta)\sigma_B}\right]n_tg_t$$

$$-\frac{2}{\gamma_\upsilon\sigma_C}\left[\frac{\Theta_1\tau}{2(1-\beta)\sigma_B}+\frac{\Theta_1\tau}{2(1-\beta)\sigma_B}-\frac{\Theta_2}{2}-\frac{\Theta_2}{2}+\frac{\Theta_2\gamma_\upsilon\sigma_C(1+\varphi)}{2}\right]n_ta_t$$

$$+\frac{(1-\upsilon)\Theta_1}{(1-\beta)\sigma_B\gamma_\upsilon\sigma_C}n_t\hat{\zeta}_t$$

$$
\begin{aligned}
&= \frac{1}{(\gamma_v \sigma_C)^2}\left[-\frac{\Theta_1 \tau \gamma_v \sigma_C}{(1-\beta)\sigma_B} + \Theta_2 \gamma_v \sigma_C\right] n_t^2 \\
&- \frac{2}{\gamma_v^2 \sigma_C}\left[\begin{array}{l} \dfrac{\tau\Theta_1[\eta\upsilon(2-\upsilon)-\gamma_v]\gamma_v\sigma_C}{2(1-\upsilon)(1-\beta)\sigma_B} \\ -\dfrac{\eta\upsilon(2-\upsilon)\sigma_C\gamma_v\Theta_2}{2(1-\alpha)} + \dfrac{\gamma_v\Theta_1}{2} - \dfrac{\upsilon[\eta\upsilon(2-\upsilon)-\gamma_v]\gamma_v\sigma_C\Theta_2}{2(1-\upsilon)(1-\alpha)} \end{array}\right] n_t z_t \\
&- \frac{2\upsilon}{\gamma_v}\left[-\frac{\tau\Theta_1}{(1-\beta)\sigma_B 2} - \frac{\eta\Theta_3 - \Theta_2}{2}\right] n_t z_{1,t}^* \\
&- \frac{2\upsilon}{\gamma_v}\left[-\frac{\tau\Theta_1[\eta\upsilon(2-\upsilon)-\gamma_v]}{(1-\upsilon)2(1-\beta)\sigma_B} - \frac{\tau\Theta_1(1-\alpha)\eta(2-\upsilon)}{2(1-\beta)\sigma_B} + \frac{\eta(2-\upsilon)\Theta_2}{2} + \frac{[\eta\upsilon(2-\upsilon)-\gamma_v]\Theta_2}{(1-\upsilon)2}\right] n_t z_{2,t}^* \\
&- \frac{2\sigma_G}{\gamma_v\sigma_C}\left[+\frac{\Theta_2}{2} - \frac{\Theta_1(1+\tau)}{2(1-\beta)\sigma_B}\right] n_t g_t \\
&- \frac{2}{\gamma_v\sigma_C}\left[\frac{\Theta_1\tau}{(1-\beta)\sigma_B} + \frac{\alpha\Theta_2}{2} + \frac{\Theta_2\gamma_v\sigma_C(1+\varphi)}{2}\right] n_t a_t \\
&+ \frac{(1-\upsilon)\Theta_1}{(1-\beta)\sigma_B\gamma_v\sigma_C} n_t \hat{\zeta}_t
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(\gamma_v \sigma_C)^2}\left[-\frac{\Theta_1 \tau \gamma_v \sigma_C}{(1-\beta)\sigma_B} + \Theta_2 \gamma_v \sigma_C\right] n_t^2 \\
&- \frac{2}{\gamma_v^2 \sigma_C}\left[\frac{\tau\Theta_1[\eta\upsilon(2-\upsilon)-\gamma_v]\gamma_v\sigma_C}{2(1-\upsilon)(1-\beta)\sigma_B} - \frac{\upsilon\sigma_C\gamma_v\Theta_2\eta(2-\upsilon)}{2(1-\alpha)} + \frac{\gamma_v\Theta_1}{2} - \frac{\upsilon\gamma_v\sigma_C\Theta_2[\eta\upsilon(2-\upsilon)-\gamma_v]}{2(1-\upsilon)(1-\alpha)}\right] n_t z_t \\
&- \frac{2\upsilon}{\gamma_v^2\sigma_C}\left[-\frac{\sigma_C\gamma_v\tau\Theta_1}{(1-\beta)\sigma_B 2} - \frac{\sigma_C\gamma_v[\eta\Theta_3 - \Theta_2]}{(1-\alpha)2}\right] n_t z_{1,t}^* \\
&- \frac{2\upsilon}{\gamma_v^2\sigma_C}\left[\begin{array}{l} -\dfrac{\tau\Theta_1\gamma_v\sigma_C[\eta\upsilon(2-\upsilon)-\gamma_v]}{(1-\upsilon)2(1-\beta)\sigma_B} - \dfrac{\tau\Theta_1\eta\gamma_v\sigma_C(2-\upsilon)}{2(1-\beta)\sigma_B} + \dfrac{\eta(2-\upsilon)\gamma_v\sigma_C\Theta_2}{(1-\alpha)2} \\ +\dfrac{[\eta\upsilon(2-\upsilon)-\gamma_v]\gamma_v\sigma_C\Theta_2}{(1-\alpha)(1-\upsilon)2} \end{array}\right] n_t z_{2,t}^* \\
&- \frac{2\sigma_G}{(\gamma_v\sigma_C)^2}\left[+\frac{\gamma_v\sigma_C\Theta_2}{(1-\alpha)2} - \frac{\gamma_v\sigma_C\Theta_1(1+\tau)}{2(1-\beta)\sigma_B}\right] n_t g_t \\
&- \frac{2}{(\gamma_v\sigma_C)^2}\left[\frac{\gamma_v\sigma_C\Theta_1\tau}{(1-\beta)\sigma_B} + \frac{\Theta_2\gamma_v\sigma_C\gamma_v\sigma_C(1+\varphi)}{2(1-\alpha)}\right] n_t a_t \\
&+ \frac{(1-\upsilon)\Theta_1}{(1-\beta)\sigma_B\gamma_v\sigma_C} n_t \hat{\zeta}_t
\end{aligned}
$$

(4-13-35b)

Combinig Eqs.(4-13-35a) and (4-13-35b) yields:

$$\frac{1}{(\gamma_v\sigma_C)^2}\left\{v^2\Gamma_{11}+(1-v)^2\Gamma_{13}+\Gamma_{14}+(\gamma_v\sigma_C)^2\Gamma_{12}+v(1-v)\Gamma_{15}-\frac{\Theta_1\tau\gamma_v\sigma_C}{(1-\beta)\sigma_B}+\Theta_2\gamma_v\sigma_C\right\}n_t^2$$

$$-\frac{2}{\gamma_v^2\sigma_C}\left\{\begin{array}{l}-\frac{v^2[\eta v(2-v)-\gamma_v]\Gamma_{11}}{1-v}-v\eta(2-v)(1-v)\Gamma_{13}-\frac{[\eta v(2-v)-\gamma_v]\Gamma_{14}}{1-v}\\-\frac{v[2v\eta(2-v)-\gamma_v]\Gamma_{15}}{2}+\frac{\tau\Theta_1[\eta v(2-v)-\gamma_v]\gamma_v\sigma_C}{2(1-v)(1-\beta)\sigma_B}-\frac{v\sigma_C\gamma_v\Theta_2\eta(2-v)}{2}\\+\frac{\gamma_v\Theta_1}{2}-\frac{v\gamma_v\sigma_C\Theta_2[\eta v(2-v)-\gamma_v]}{2(1-v)}\end{array}\right\}n_t z_t$$

$$-\frac{2v}{\gamma_v^2\sigma_C}\left[v^2\Gamma_{11}+(1-v)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}+v(1-v)\Gamma_{15}-\frac{\sigma_C\gamma_v\tau\Theta_1}{(1-\beta)\sigma_B 2}-\frac{\sigma_C\gamma_v[\eta\Theta_3-\Theta_2]}{2}\right]n_t z_{1,t}^*$$

$$-\frac{2}{\gamma_v^2\sigma_C}\left[\begin{array}{l}\frac{v^2[\eta v(2-v)-\gamma_v]\Gamma_{11}}{(1-v)}+v\eta(2-v)(1-v)\Gamma_{13}+\frac{[\eta v(2-v)-\gamma_v]\Gamma_{14}}{(1-v)}\\+\frac{v[2v\eta(2-v)-\gamma_v]\Gamma_{15}}{2}-\frac{\tau\Theta_1\gamma_v\sigma_C[\eta v(2-v)-\gamma_v]}{(1-v)2(1-\beta)\sigma_B}-\frac{\tau\Theta_1\eta\gamma_v\sigma_C(2-v)}{2(1-\beta)\sigma_B}\\+\frac{\eta(2-v)\gamma_v\sigma_C\Theta_2}{2}+\frac{[\eta v(2-v)-\gamma_v]\gamma_v\sigma_C\Theta_2}{(1-v)2}\end{array}\right]n_t z_{2,t}^*$$

$$-\frac{2\sigma_G}{(\gamma_v\sigma_C)^2}\left[v^2\Gamma_{11}+v(1-v)^2\Gamma_{13}+\frac{\Gamma_{14}}{2}+v(1-v)\Gamma_{15}+\frac{\gamma_v\sigma_C\Theta_2}{(1-\alpha)2}-\frac{\gamma_v\sigma_C\Theta_1(1+\tau)}{2(1-\beta)\sigma_B}\right]n_t g_t$$

$$-\frac{2}{(\gamma_v\sigma_C)^2}\left[\begin{array}{l}-v^2\Gamma_{11}-(\gamma_v\sigma_C)^2\Gamma_{12}-(1-v)^2\Gamma_{13}-\frac{\Gamma_{14}}{2}-v(1-v)\Gamma_{15}+\frac{\gamma_v\sigma_C\Theta_1\tau}{(1-\beta)\sigma_B}\\+\frac{\Theta_2(\gamma_v\sigma_C)^2(1+\varphi)}{2}\end{array}\right]n_t a_t$$

$$+\frac{(1-v)\Theta_1}{(1-\beta)\sigma_B\gamma_v\sigma_C}n_t\hat{\zeta}_t$$

$$
\begin{aligned}
&= \frac{1}{(\gamma_v \sigma_C)^2}\left(\upsilon^2 \Gamma_{11} + (1-\upsilon)^2 \Gamma_{13} + \Gamma_{14} - \frac{\Theta_1 \tau \gamma_v \sigma_C}{(1-\beta)\sigma_B} + \Theta_2 \gamma_v \sigma_C + (\gamma_v \sigma_C)^2 \Gamma_{12} + \upsilon(1-\upsilon)\Gamma_{15} \right) n_t^2 \\
&- \frac{2}{\gamma_v^2 \sigma_C}\left\{ \begin{array}{l} \dfrac{\begin{array}{l}[\eta\upsilon(2-\upsilon)-\gamma_v]\{[\tau\Theta_1\gamma_v\sigma_C - 2(1-\beta)\sigma_B(\upsilon^2\Gamma_{11}+\Gamma_{14})] - (1-\beta)\sigma_B\upsilon\gamma_v\sigma_C\Theta_2\} \\ -\upsilon\sigma_C\gamma_v\Theta_2\eta(1-\upsilon)(2-\upsilon)(1-\beta)\sigma_B \end{array}}{2(1-\upsilon)(1-\beta)\sigma_B} \\ + \dfrac{[\gamma_v\Theta_1 - \upsilon\eta(2-\upsilon)(1-\upsilon)2\Gamma_{13} - \upsilon[2\upsilon\eta(2-\upsilon)-\gamma_v]\Gamma_{15}]}{2} \end{array} \right\} n_t z_t \\
&- \frac{2\upsilon}{\gamma_v^2 \sigma_C}\left\{ \upsilon^2\Gamma_{11} + (1-\upsilon)^2\Gamma_{13} + \frac{\Gamma_{14}}{2} + \upsilon(1-\upsilon)\Gamma_{15} - \frac{\sigma_C\gamma_v\tau\Theta_1}{(1-\beta)\sigma_B 2} - \frac{\sigma_C\gamma_v[\eta\Theta_3 - \Theta_2]}{2} \right\} n_t z_{1,t}^* \\
&- \frac{2}{\gamma_v^2 \sigma_C}\left\{ \begin{array}{l} \dfrac{\upsilon^2[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{11}}{(1-\upsilon)} + \dfrac{\upsilon\eta(2-\upsilon)(1-\upsilon)^2\Gamma_{13}}{(1-\upsilon)} + \dfrac{[\eta\upsilon(2-\upsilon)-\gamma_v]\Gamma_{14}}{(1-\upsilon)} \\ + \dfrac{\upsilon[2\upsilon\eta(2-\upsilon)-\gamma_v]\Gamma_{15}}{2} - \dfrac{\tau\Theta_1\gamma_v\sigma_C[\eta(2-\upsilon)\upsilon + \eta(2-\upsilon)(1-\upsilon) - \gamma_v]}{(1-\upsilon)2(1-\beta)\sigma_B} \\ + \dfrac{\gamma_v\sigma_C\Theta_2[\eta(2-\upsilon)(1-\upsilon) + \eta(2-\upsilon)\upsilon - \gamma_v]}{(1-\upsilon)2} \end{array} \right\} n_t z_{2,t}^* \\
&- \frac{2\sigma_G}{(\gamma_v\sigma_C)^2}\left[\upsilon^2\Gamma_{11} + \upsilon(1-\upsilon)^2\Gamma_{13} + \frac{\Gamma_{14}}{2} + \upsilon(1-\upsilon)\Gamma_{15} - \frac{\gamma_v\sigma_C\Theta_1(1+\tau)}{2(1-\beta)\sigma_B} + \frac{\gamma_v\sigma_C\Theta_2}{2} \right] n_t g_t \\
&- \frac{2}{(\gamma_v\sigma_C)^2}\left\{ \begin{array}{l} -\upsilon^2\Gamma_{11} - (\gamma_v\sigma_C)^2\Gamma_{12} - (1-\upsilon)^2\Gamma_{13} - \dfrac{\Gamma_{14}}{2} - \upsilon(1-\upsilon)\Gamma_{15} + \dfrac{\gamma_v\sigma_C\Theta_1\tau}{(1-\beta)\sigma_B} \\ + \dfrac{\Theta_2(\gamma_v\sigma_C)^2(1+\varphi)}{2} \end{array} \right\} n_t a_t \\
&+ \frac{(1-\upsilon)\Theta_1}{(1-\beta)\sigma_B\gamma_v\sigma_C} n_t \hat{\zeta}_t
\end{aligned}
$$

$$
=\frac{1}{\left(\gamma_{\upsilon}\sigma_{C}\right)^{2}}\left(\upsilon^{2}\Gamma_{11}+(1-\upsilon)^{2}\Gamma_{13}+\Gamma_{14}-\frac{\Theta_{1}\tau\gamma_{\upsilon}\sigma_{C}}{(1-\beta)\sigma_{B}}+\Theta_{2}\gamma_{\upsilon}\sigma_{C}+\left(\gamma_{\upsilon}\sigma_{C}\right)^{2}\Gamma_{12}+\upsilon(1-\upsilon)\Gamma_{15}\right)n_{t}^{2}
$$

$$
-\frac{2}{\gamma_{\upsilon}^{2}\sigma_{C}}\left\{\begin{array}{l}
+\dfrac{\left[\gamma_{\upsilon}\Theta_{1}-\upsilon\eta(2-\upsilon)(1-\upsilon)2\Gamma_{13}-\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_{\upsilon}\right]\Gamma_{15}\right]}{2}\\
+\dfrac{\left[\upsilon\eta(2-\upsilon)-\gamma_{\upsilon}\right]\left[\tau\Theta_{1}\gamma_{\upsilon}\sigma_{C}-2(1-\beta)\sigma_{B}\left(\upsilon^{2}\Gamma_{11}+\Gamma_{14}\right)\right]}{2(1-\upsilon)(1-\beta)\sigma_{B}}\\
-\dfrac{\left[\upsilon\eta(2-\upsilon)-\gamma_{\upsilon}\right]\upsilon\gamma_{\upsilon}\sigma_{C}\Theta_{2}+\upsilon\sigma_{C}\gamma_{\upsilon}\Theta_{2}\eta(1-\upsilon)(2-\upsilon)}{2(1-\upsilon)}
\end{array}\right\}n_{t}z_{t}
$$

$$
-\frac{2\upsilon}{\gamma_{\upsilon}^{2}\sigma_{C}}\left\{\upsilon^{2}\Gamma_{11}+(1-\upsilon)^{2}\Gamma_{13}+\frac{\Gamma_{14}}{2}+\upsilon(1-\upsilon)\Gamma_{15}-\frac{\sigma_{C}\gamma_{\upsilon}\tau\Theta_{1}}{(1-\beta)\sigma_{B}2}-\frac{\sigma_{C}\gamma_{\upsilon}\left[\eta(1-\alpha)\Theta_{3}-\Theta_{2}\right]}{2}\right\}n_{t}z_{1,t}^{*}
$$

$$
-\frac{2}{\gamma_{\upsilon}^{2}\sigma_{C}}\left\{\begin{array}{l}
\dfrac{\upsilon^{2}\left[\eta\upsilon(2-\upsilon)-\gamma_{\upsilon}\right]\Gamma_{11}+(1-\upsilon)^{2}\upsilon\eta(2-\upsilon)\Gamma_{13}}{(1-\upsilon)}+\dfrac{\left[\eta\upsilon(2-\upsilon)-\gamma_{\upsilon}\right]\Gamma_{14}}{(1-\upsilon)}\\
+\dfrac{\upsilon\left[2\upsilon\eta(2-\upsilon)-\gamma_{\upsilon}\right]\Gamma_{15}}{2}-\dfrac{\tau\Theta_{1}\gamma_{\upsilon}\sigma_{C}\left[\eta(2-\upsilon)-\gamma_{\upsilon}\right]}{(1-\upsilon)2(1-\beta)\sigma_{B}}\\
+\dfrac{\gamma_{\upsilon}\sigma_{C}\Theta_{2}\left\{\eta(2-\upsilon)(1-\upsilon)+\eta(2-\upsilon)\upsilon-\gamma_{\upsilon}\right\}}{(1-\upsilon)2}
\end{array}\right\}n_{t}z_{2,t}^{*}
$$

$$
-\frac{2\sigma_{G}}{\left(\gamma_{\upsilon}\sigma_{C}\right)^{2}}\left\{\upsilon^{2}\Gamma_{11}+\upsilon(1-\upsilon)^{2}\Gamma_{13}+\frac{\Gamma_{14}}{2}+\upsilon(1-\upsilon)\Gamma_{15}-\frac{\gamma_{\upsilon}\sigma_{C}\Theta_{1}(1+\tau)}{2(1-\beta)\sigma_{B}}+\frac{\gamma_{\upsilon}\sigma_{C}\Theta_{2}}{2}\right\}n_{t}g_{t}
$$

$$
-\frac{2}{\left(\gamma_{\upsilon}\sigma_{C}\right)^{2}}\left\{\begin{array}{l}
-\upsilon^{2}\Gamma_{11}-\left(\gamma_{\upsilon}\sigma_{C}\right)^{2}\Gamma_{12}-(1-\upsilon)^{2}\Gamma_{13}-\dfrac{\Gamma_{14}}{2}-\upsilon(1-\upsilon)\Gamma_{15}+\dfrac{\gamma_{\upsilon}\sigma_{C}\Theta_{1}\tau}{(1-\beta)\sigma_{B}}\\
+\dfrac{\Theta_{2}\left(\gamma_{\upsilon}\sigma_{C}\right)^{2}(1+\varphi)}{2}
\end{array}\right\}n_{t}a_{t}
$$

$$
+\frac{(1-\upsilon)\Theta_{1}}{(1-\beta)\sigma_{B}\gamma_{\upsilon}\sigma_{C}}n_{t}\hat{\zeta}_{t}
$$

.

Let define:

$$
\Gamma_{51}\equiv\frac{\Theta_{1}\tau\gamma_{\upsilon}\sigma_{C}}{(1-\beta)\sigma_{B}},
$$

$$
\Gamma_{52}\equiv\upsilon^{2}\Gamma_{11}+(1-\upsilon)^{2}\Gamma_{13}+\Gamma_{14}-\Gamma_{51}+\Theta_{2}\gamma_{\upsilon}\sigma_{C}
$$

$$
\Gamma_{53}\equiv\left(\gamma_{\upsilon}\sigma_{C}\right)^{2}\Gamma_{12}+\upsilon(1-\upsilon)\Gamma_{15}
$$

$$\Gamma_{54} \equiv \gamma_v \Theta_1 - \upsilon\eta(2-\upsilon)(1-\upsilon)2\Gamma_{13} - \upsilon\left[2\upsilon\eta(2-\upsilon) - \gamma_v\right]\Gamma_{15}$$

$$\Gamma_{55} \equiv \eta\upsilon(2-\upsilon) - \gamma_v$$

$$\Gamma_{56} \equiv 2\upsilon\eta(2-\upsilon) - \gamma_v$$

$$\Gamma_{57} \equiv \gamma_v \Theta_1 - \upsilon\eta(2-\upsilon)(1-\upsilon)2\Gamma_{13} - \upsilon\Gamma_{56}\Gamma_{15}$$

$$\Gamma_{57B} \equiv \tau\Theta_1\gamma_v\sigma_C - 2(1-\beta)\sigma_B\left(\upsilon^2\Gamma_{11} + \Gamma_{14}\right)$$

$$\Gamma_{58} \equiv \frac{\Gamma_{57}(1-\upsilon)(1-\beta)\sigma_B + \Gamma_{55}\Gamma_{57B}}{2(1-\upsilon)(1-\beta)\sigma_B}$$

$$\Gamma_{59} \equiv \frac{\Gamma_{55}\upsilon\gamma_v\sigma_C\Theta_2 + \upsilon\sigma_C\gamma_v\Theta_2\eta(1-\upsilon)(2-\upsilon)}{2(1-\upsilon)}$$

$$\Gamma_{60} \equiv \upsilon^2\Gamma_{11} + (1-\upsilon)^2\Gamma_{13} + \frac{\Gamma_{14}}{2} + \upsilon(1-\upsilon)\Gamma_{15} - \frac{\Gamma_{51}}{2}$$

$$\Gamma_{61} \equiv \frac{\sigma_C\gamma_v(\eta\Theta_3 - \Theta_2)}{2}$$

$$\Gamma_{62} \equiv \frac{\Gamma_{55}\left(\upsilon^2\Gamma_{11} + \Gamma_{14}\right) + (1-\upsilon)^2\upsilon\eta(2-\upsilon)\Gamma_{13}}{(1-\upsilon)}$$

$$\Gamma_{63} \equiv \tau\Theta_1\gamma_v\sigma_C\left[\eta(2-\upsilon) - \gamma_v\right]$$

$$\Gamma_{64} \equiv \frac{(1-\upsilon)(1-\beta)\sigma_B(2\Gamma_{62} + \upsilon\Gamma_{56}\Gamma_{15}) - \Gamma_{63}}{2(1-\upsilon)(1-\beta)\sigma_B}$$

$$\Gamma_{65} \equiv \frac{\gamma_v\sigma_C\Theta_2\left[\eta(2-\upsilon) - \gamma_v\right]}{(1-\upsilon)2}$$

$$\Gamma_{66} \equiv \frac{\gamma_v\sigma_C\Theta_1(1+\tau)}{2(1-\beta)\sigma_B}$$

$$\Gamma_{67} \equiv \upsilon^2\Gamma_{11} + \upsilon(1-\upsilon)^2\Gamma_{13} + \frac{\Gamma_{14}}{2} + \upsilon(1-\upsilon)\Gamma_{15} - \Gamma_{66}$$

$$\Gamma_{68} \equiv \frac{\gamma_v\sigma_C\Theta_2}{2}$$

$$\Gamma_{69} \equiv \Gamma_{51} - \upsilon^2\Gamma_{11} - (\gamma_v\sigma_C)^2\Gamma_{12} - (1-\upsilon)^2\Gamma_{13} - \frac{\Gamma_{14}}{2} - \upsilon(1-\upsilon)\Gamma_{15}$$

$$\Gamma_{70} \equiv \frac{\Theta_2(\gamma_v\sigma_C)^2\sigma_C(1+\varphi)}{2}$$

$$\Gamma_{71} \equiv -\frac{(1-\upsilon)\Theta_1}{(1-\beta)\sigma_B 2}$$

Then, the previous expression can be rewritten as:

$$\begin{aligned}&\frac{1}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{52}+\Gamma_{53}]n_t^2-\frac{2}{\gamma_\upsilon^2\sigma_C}[\Gamma_{58}-\Gamma_{59}]n_t z_t-\frac{2\upsilon}{\gamma_\upsilon^2\sigma_C}[\Gamma_{60}-\Gamma_{61}]n_t z_{1,t}^*\\&-\frac{2}{\gamma_\upsilon^2\sigma_C}[\Gamma_{64}+\Gamma_{65}]n_t z_{2,t}^*-\frac{2\sigma_G}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{67}+\Gamma_{68}]n_t g_t-\frac{2}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{69}+\Gamma_{70}]n_t a_t\\&-\frac{2}{\gamma_\upsilon\sigma_C}\Gamma_{71}n_t\hat{\zeta}_t\end{aligned} \quad \text{. (4-13-35c)}$$

Plugging Eq. (4-13-35c) into Eq.(4-13-35) yields:

$$\begin{aligned}&\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\Phi}{(1-\alpha)\sigma_C}y_t-\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}\\&=\Theta_1(1-\beta)^{-1}\bar{\omega}_H+\Theta_2\kappa^{-1}\bar{\nu}_H+\Theta_8\kappa_W^{-1}\bar{\nu}_H^W\\&+\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\begin{aligned}&\frac{1}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{52}+\Gamma_{53}]n_t^2-\frac{2}{\gamma_\upsilon^2\sigma_C}[\Gamma_{58}-\Gamma_{59}]n_t z_t-\frac{2\upsilon}{\gamma_\upsilon^2\sigma_C}[\Gamma_{60}-\Gamma_{61}]n_t z_{1,t}^*\\&-\frac{2}{\gamma_\upsilon^2\sigma_C}[\Gamma_{64}+\Gamma_{65}]n_t z_{2,t}^*-\frac{2\sigma_G}{(\gamma_\upsilon\sigma_C)^2}[(1-\alpha)\Gamma_{67}+\Gamma_{68}]n_t g_t\\&-\frac{2}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{69}+\Gamma_{70}]n_t a_t-\frac{2}{\gamma_\upsilon\sigma_C}\Gamma_{71}n_t\hat{\zeta}_t+\frac{\Theta_2\varphi(2+\varphi)}{2}n_t^2\\&+\frac{\Theta_2\varepsilon_p[\kappa_p+\varphi]}{2\kappa_p}\pi_{H,t}^2+\frac{\Theta_2\varepsilon_w(1+\varphi)}{2}(\pi_{H,t}^w)^2\end{aligned}\right\},\\&+\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}+o\left(\|\xi\|^2\right)\end{aligned}$$

which can be rewritten as:

$$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left\{\frac{\Phi}{(1-\alpha)\sigma_C}y_t-\frac{\upsilon\eta(2-\upsilon)}{1-\upsilon}s_t\right\}$$

$$=\Theta_1(1-\beta)^{-1}\bar{\omega}_H+\Theta_2\kappa^{-1}\bar{\nu}_H+\Theta_8\kappa_W^{-1}\bar{\nu}_H^W$$

$$+\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\begin{array}{l}\frac{1}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{52}+\Gamma_{53}+\Gamma_{72}]n_t^2-\frac{2}{\gamma_\upsilon^2\sigma_C}[\Gamma_{58}-\Gamma_{59}]n_t z_t \\ -\frac{2\upsilon}{\gamma_\upsilon^2\sigma_C}[\Gamma_{60}-\Gamma_{61}]n_t z_{1,t}^*-\frac{2}{\gamma_\upsilon^2\sigma_C}[\Gamma_{64}+\Gamma_{65}]n_t z_{2,t}^* \\ -\frac{2\sigma_G}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{67}+\Gamma_{68}]n_t g_t-\frac{2}{(\gamma_\upsilon\sigma_C)^2}[\Gamma_{69}+\Gamma_{70}]n_t a_t-\frac{2}{\gamma_\upsilon\sigma_C}\Gamma_{71}n_t\hat{\zeta}_t \\ +\frac{\Theta_2\varepsilon_p[\kappa_p+\varphi]}{2\kappa_p}\pi_{H,t}^2+\frac{\Theta_2\varepsilon_w(1+\varphi)}{2}(\pi_{H,t}^w)^2\end{array}\right), \quad (4\text{-}13\text{-}82)$$

$$+\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}$$

$$+o\left(\|\xi\|^2\right)$$

with $\Gamma_{72}\equiv\frac{\Theta_2\varphi(2+\varphi)(\gamma_\upsilon\sigma_C)^2}{2}$.

Plugging into Eq.(4-13-82) into Eq.(4-2-17) yields:

$$\tilde{W} = -\sum_{k=0}^{\infty} \beta^t \mathrm{E}_t \left\{ \begin{bmatrix} \left(\begin{array}{l} \frac{1}{(\gamma_\upsilon \sigma_C)^2}[-\Gamma_{52} - \Gamma_{53} - \Gamma_{72}] n_t^2 \\ - \left[\begin{array}{l} \frac{2}{\gamma_\upsilon^2 \sigma_C}[-\Gamma_{58} + \Gamma_{59}] n_t z_t \\ + \frac{2\upsilon}{\gamma_\upsilon^2 \sigma_C}[-\Gamma_{60} + \Gamma_{61}] n_t z_{1,t}^* \\ + \frac{2}{\gamma_\upsilon^2 \sigma_C}[-\Gamma_{64} - \Gamma_{65}] n_t z_{2,t}^* \\ + \frac{2\sigma_G}{(\gamma_\upsilon \sigma_C)^2}[-\Gamma_{67} - \Gamma_{68}] n_t g_t \\ + \frac{2}{(\gamma_\upsilon \sigma_C)^2}[-\Gamma_{69} - \Gamma_{70}] n_t a_t \\ + \frac{2\Gamma_{71}(-1)}{\gamma_\upsilon \sigma_C} n_t \hat{\zeta}_t \end{array} \right] \end{array} \right) \\ - \frac{\Theta_2 \varepsilon_p [\kappa_p + \varphi]}{(1-\alpha) 2\kappa_p} \pi_{H,t}^2 - \frac{\Theta_2 \varepsilon_w (1+\varphi)}{2} \left(\pi_{H,t}^w\right)^2 \\ + \frac{(1-\Phi)(1+\varphi)}{2\sigma_C} n_{H,t}^2 \\ + \frac{\varepsilon_p (1-\Phi)}{2\kappa_p (1-\alpha)\sigma_C} \pi_{H,t}^2 + \frac{\varepsilon_w (1+\varepsilon_w \varphi)(1-\Phi)}{2\kappa_w \sigma_C} \left(\pi_t^W\right)^2 \end{bmatrix} \right\} + o\left(\|\xi\|^3\right),$$

which can be rewritten as:

$$\tilde{\mathcal{W}} = -\sum_{k=0}^{\infty}\beta^t \mathrm{E}_t \left\{ \begin{array}{l} \left(\begin{array}{l} \dfrac{1}{(\gamma_\upsilon \sigma_C)^2}\left[\begin{array}{l} -\Gamma_{52} - \Gamma_{53} - \Gamma_{72} \\ + \dfrac{\gamma_\upsilon^2 \sigma_C (1-\Phi)(1+\varphi)}{2} \end{array}\right] n_t^2 \\ - \left[\begin{array}{l} \dfrac{2}{\gamma_\upsilon^2 \sigma_C}[-\Gamma_{58} + \Gamma_{59}] n_t z_t + \dfrac{2\upsilon}{\gamma_\upsilon^2 \sigma_C}[-\Gamma_{60} + \Gamma_{61}] n_t z_{1,t}^* \\ + \dfrac{2}{\gamma_\upsilon^2 \sigma_C}[-\Gamma_{64} - \Gamma_{65}] n_t z_{2,t}^* + \dfrac{2\sigma_G}{(\gamma_\upsilon \sigma_C)^2}[-\Gamma_{67} - \Gamma_{68}] n_t g_t \\ + \dfrac{2}{(\gamma_\upsilon \sigma_C)^2}[-\Gamma_{69} - \Gamma_{70}] n_t a_t + \dfrac{2(-1)}{\gamma_\upsilon \sigma_C}\Gamma_{71} n_t \hat{\zeta}_t \end{array} \right] \end{array} \right) \\ + \dfrac{\varepsilon_p \left\{(1-\Phi) - \Theta_2 \sigma_C \left[\kappa_p + \varphi\right]\right\}}{2\kappa_p \sigma_C} \pi_{H,t}^2 \\ + \dfrac{\varepsilon_w \left[-\Theta_2 (1+\varphi)\kappa_w \sigma_C + (1+\varepsilon_w \varphi)(1-\Phi)\right]}{2\kappa_w \sigma_C} \left(\pi_t^W\right)^2 \end{array} \right\} + o\left(\|\xi\|^3\right),$$

In the previous expression, lines 1 and 2 can be rewritten as:

$$\frac{1}{(\gamma_v\sigma_C)^2}\begin{bmatrix}-\Gamma_{52}-\Gamma_{53}-\Gamma_{72}\\+\frac{\gamma_v^2\sigma_C(1-\Phi)(1+\varphi)}{2}\end{bmatrix}n_t^2-\begin{bmatrix}\frac{2}{\gamma_v^2\sigma_C}[-\Gamma_{58}+\Gamma_{59}]n_tz_t\\+\frac{2\upsilon}{\gamma_v^2\sigma_C}[-\Gamma_{60}+\Gamma_{61}]n_tz_{1,t}^*\\+\frac{2}{\gamma_v^2\sigma_C}[-\Gamma_{64}-\Gamma_{65}]n_tz_{2,t}^*\\+\frac{2\sigma_G}{(\gamma_v\sigma_C)^2}[-\Gamma_{67}-\Gamma_{68}]n_tg_t\\+\frac{2}{(\gamma_v\sigma_C)^2}[-\Gamma_{69}-\Gamma_{70}]n_ta_t\\+\frac{2(-1)}{\gamma_v\sigma_C}\Gamma_{71}n_t\hat{\zeta}_t\end{bmatrix}$$

$$=\frac{1}{(\gamma_v\sigma_C)^2}\begin{bmatrix}-\Gamma_{52}-\Gamma_{53}-\Gamma_{72}\\+\frac{\gamma_v^2\sigma_C(1-\Phi)(1+\varphi)}{2}\end{bmatrix}n_t^2-2n_t\begin{bmatrix}\frac{(-\Gamma_{58}+\Gamma_{59})}{\gamma_v^2\sigma_C}z_t\\+\frac{\upsilon[-\Gamma_{60}+\Gamma_{61}]}{\gamma_v^2\sigma_C}z_{1,t}^*\\+\frac{(-\Gamma_{64}-\Gamma_{65})}{\gamma_v^2\sigma_C}z_{2,t}^*\\+\frac{\sigma_G[-\Gamma_{67}-\Gamma_{68}]}{(\gamma_v\sigma_C)^2}g_t\\+\frac{(-\Gamma_{69}-\Gamma_{70})}{(\gamma_v\sigma_C)^2}a_t\\+\frac{(-1)}{\gamma_v\sigma_C}\Gamma_{71}\hat{\zeta}_t\end{bmatrix}.$$

Let define $\Omega_0\equiv-\Gamma_{52}-\Gamma_{53}-\Gamma_{72}+\frac{\gamma_v^2\sigma_C(1-\Phi)(1+\varphi)}{2}$. Then:

$$\frac{\Omega_0}{(\gamma_v\sigma_C)^2}n_t^2-2n_t\begin{bmatrix}\frac{(-\Gamma_{58}+\Gamma_{59})}{\gamma_v^2\sigma_C}z_t\\+\frac{\upsilon(-\Gamma_{60}+\Gamma_{61})}{\gamma_v^2\sigma_C}z_{1,t}^*\\+\frac{(-\Gamma_{64}-\Gamma_{65})}{\gamma_v^2\sigma_C}z_{2,t}^*\\+\frac{\sigma_G(-\Gamma_{67}-\Gamma_{68})}{(\gamma_v\sigma_C)^2}g_t\\+\frac{(-\Gamma_{69}-\Gamma_{70})}{(\gamma_v\sigma_C)^2}a_t\\+\frac{(-1)}{\gamma_v\sigma_C}\Gamma_{71}\hat{\zeta}_t\end{bmatrix}=\frac{\Omega_0}{(\gamma_v\sigma_C)^2}\left\{n_t^2-2\frac{(\gamma_v\sigma_C)^2}{\Omega_0}n_t\begin{bmatrix}\frac{(-\Gamma_{58}+\Gamma_{59})}{\gamma_v^2\sigma_C}z_t\\+\frac{\upsilon(-\Gamma_{60}+\Gamma_{61})}{\gamma_v^2\sigma_C}z_{1,t}^*\\+\frac{(-\Gamma_{64}-\Gamma_{65})}{\gamma_v^2\sigma_C}z_{2,t}^*\\+\frac{\sigma_G(-\Gamma_{67}-\Gamma_{68})}{(\gamma_v\sigma_C)^2}g_t\\+\frac{(-\Gamma_{69}-\Gamma_{70})}{(\gamma_v\sigma_C)^2}a_t\\+\frac{(-1)}{\gamma_v\sigma_C}\Gamma_{71}\hat{\zeta}_t\end{bmatrix}\right\}$$

,

$$=\frac{\Omega_0}{(\gamma_v\sigma_C)^2}\left\{n_t^2-2n_t\begin{bmatrix}\frac{\sigma_C(-\Gamma_{58}+\Gamma_{59})}{\Omega_0}z_t\\+\frac{\upsilon(-\Gamma_{60}+\Gamma_{61})\sigma_C}{\Omega_0}z_{1,t}^*\\+\frac{(-\Gamma_{64}-\Gamma_{65})\sigma_C}{\Omega_0}z_{2,t}^*\\+\frac{\sigma_G(-\Gamma_{67}-\Gamma_{68})}{\Omega_0}g_t\\+\frac{(-\Gamma_{69}-\Gamma_{70})}{\Omega_0}a_t\\+\frac{\gamma_v\sigma_C(-1)}{\Omega_0}\Gamma_{71}\hat{\zeta}_t\end{bmatrix}\right\}$$

Let define:

$$\Omega_1\equiv\frac{\sigma_C(-\Gamma_{58}+\Gamma_{59})}{\Omega_0}$$

$$\Omega_2\equiv\frac{(-\Gamma_{60}+\Gamma_{61})\sigma_C}{\Omega_0}$$

$$\Omega_3\equiv\frac{(-\Gamma_{64}-\Gamma_{65})\sigma_C}{\Omega_0}$$

$$\Omega_4\equiv\frac{\sigma_G(-\Gamma_{67}-\Gamma_{68})}{\Omega_0}$$

$$\Omega_5 \equiv \frac{(-\Gamma_{69} - \Gamma_{70})}{\Omega_0}$$

$$\Omega_6 \equiv \frac{\gamma_\upsilon \sigma_C (-1)}{\Omega_0} \Gamma_{71}$$

and let define:

$$n_t^e = \Omega_1 z_t + \upsilon \Omega_2 z_{1,t}^* + \Omega_3 z_{2,t}^* + \Omega_4 g_t + \Omega_5 a_t + \Omega_6 \hat{\zeta}_t .$$

Finally, we have:

$$\tilde{\mathcal{W}} = -\sum_{k=0}^{\infty} \beta^t \mathrm{E}_t \left\{ \begin{array}{l} \frac{\Omega_0}{(\gamma_\upsilon \sigma_C)^2} \left(n_t - n_t^e\right)^2 \\ + \frac{\varepsilon_p \left[(1-\Phi) - \Theta_2 \sigma_C \left(\kappa_p + \varphi\right)\right]}{2\kappa_p \sigma_C} \pi_{H,t}^2 \\ + \frac{\varepsilon_w \left[-\Theta_2 (1+\varphi) \kappa_w \sigma_C + (1+\varepsilon_w \varphi)(1-\Phi)\right]}{2\kappa_w \sigma_C} \left(\pi_t^W\right)^2 \end{array} \right\} + \text{s.o.t.i.p.} + o\left(\|\xi\|^3\right).$$

Let define $\Lambda_n^F \equiv \frac{2\Omega_0^F}{(\gamma_\upsilon \sigma_C)^2}$, $\Lambda_p^F \equiv \frac{\varepsilon_p \left[(1-\Phi) - \Theta_2 \sigma_C \left(\kappa_p + \varphi\right)\right]}{\kappa_p \sigma_C}$ and

$\Lambda_w^F \equiv \frac{\varepsilon_w \left[-\Theta_2 (1+\varphi) \kappa_w \sigma_C + (1+\varepsilon_w \varphi)(1-\Phi)\right]}{\kappa_w \sigma_C}$. Then we have:

$$\tilde{\mathcal{W}} = -\sum_{k=0}^{\infty} \beta^t \mathrm{E}_t \left[\frac{\Lambda_n}{2} \hat{n}_t^2 + \frac{\Lambda_P}{2} \pi_{H,t}^2 + \frac{\Lambda_w}{2} \left(\pi_t^W\right)^2 \right] + \text{s.o.t.i.p.} + o\left(\|\xi\|^3\right).$$

Based on the previous expression, we have:

$$\mathcal{L}^F \sim \frac{\Lambda_n}{2} \mathrm{var}(\hat{n}_t) + \frac{\Lambda_P}{2} \mathrm{var}(\pi_{H,t}) + \frac{\Lambda_w}{2} \mathrm{var}(\pi_t^W),$$

which is Eq.(42) in the text.

6 The NKPC and the Wage PC with Efficient Level

Eq.(3-1-34) can be rewritten as:

$$\begin{aligned}\mu_t^w &= w_t - p_{H,t} - \upsilon s_t - \varphi n_t - c_t \\ &= w_t - p_t + p_t - p_{H,t} - \upsilon s_t - \varphi n_t - c_t \\ &= \omega_t - \left(p_{H,t} - p_t\right) - \upsilon s_t - \varphi n_t - c_t \\ &= \omega_t - \varphi n_t - c_t\end{aligned}, \text{ (6-0)}$$

with $\omega_t \equiv d\frac{W_t}{P_t}\Big/\frac{W}{P}$ being the real (consumption) wage. Under the flexible wage equilibrium, $\mu_t^W = 0$ is applied. Under that equilibrium, Eq.(6-0) can be rewritten as:

$\omega_t^e = \varphi n_t^e + c_t^e$, (6-1)

with $\omega_t^e \equiv \omega_t - \hat{\omega}_t$ and $c_t^e \equiv c_t - \hat{c}_t$.

Eq.(3-1-24) can be rewritten as:

$$\begin{aligned}mc_t &= \frac{1}{1-\tau}\tau_t + w_t - p_{H,t} - p_t + p_t - a_t - \frac{1}{1-\tau}\tau \\ &= \frac{1}{1-\tau}\tau_t + \omega_t - \left(p_{H,t} - p_t\right) - a_t - \frac{1}{1-\tau}\tau \\ &= \frac{1}{1-\tau}\tau_t + \omega_t - x_{H,t} - a_t - \frac{1}{1-\tau}\tau \\ &= \frac{1}{1-\tau}\tau_t + \omega_t + \upsilon s_t - a_t - \frac{1}{1-\tau}\tau\end{aligned}, \text{ (6-2)}$$

where we use Eq.(3-1-7).

with $s_t^e \equiv s_t - \hat{s}_t$.

Eq.(3-1-11) can be rewritten as:

$$s_t = \frac{1}{1-\upsilon}c_t - \frac{1}{1-\upsilon}z_t + \frac{1}{1-\upsilon}z_{2,t}^*.$$

which can be rewritten as:

$$s_t^e = \frac{1}{1-\upsilon}c_t^e - \frac{1}{1-\upsilon}z_t + \frac{1}{1-\upsilon}z_{2,t}^*. \text{ (6-4)}$$

under the efficient equilibrium.

Eq.(3-1-8) can be rewritten as:

$$c_t = \frac{1}{(1-\upsilon)\sigma_C} y_t - \frac{\eta\upsilon(2-\upsilon)}{1-\upsilon} s_t - \frac{\upsilon}{1-\upsilon} z_{1,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C} g_t$$
$$= \frac{1}{(1-\upsilon)\sigma_C} y_t - \frac{\eta\upsilon(2-\upsilon)}{1-\upsilon}\left(\frac{1}{1-\upsilon}c_t - \frac{1}{1-\upsilon}z_t + \frac{1}{1-\upsilon}z_{2,t}^*\right) - \frac{\upsilon}{1-\upsilon} z_{1,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C} g_t ,$$
$$= \frac{1}{(1-\upsilon)\sigma_C} y_t - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} c_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z_t - \frac{\upsilon}{1-\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z_{2,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C} g_t$$

where we use Eq.(3-1-11). The previous expression can be rewritten as:

$$\frac{(1-\upsilon)^2 + \eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} c_t = \frac{1}{(1-\upsilon)\sigma_C} y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z_t - \frac{\upsilon}{1-\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z_{2,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C} g_t .$$

Then, we have:

$$c_t = \frac{(1-\upsilon)^2}{\gamma_\upsilon}\left[\frac{1}{(1-\upsilon)\sigma_C} y_t + \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z_t - \frac{\upsilon}{1-\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{(1-\upsilon)^2} z_{2,t}^* - \frac{\sigma_G}{(1-\upsilon)\sigma_C} g_t\right]$$
$$= \frac{1-\upsilon}{\gamma_\upsilon\sigma_C} y_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t ,$$

with $\gamma_\upsilon \equiv (1-\upsilon)^2 + \eta\upsilon(2-\upsilon)$.

The previous expression can be rewritten as:

$$c_t^e = \frac{1-\upsilon}{\gamma_\upsilon\sigma_C} y_t^e + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t , \quad (6\text{-}5)$$

under the efficient equilibrium.

Eq.(3-1-26) can be rewritten as:

$$y_t^e = n_t^e + a_t , \quad (6\text{-}6)$$

under the efficient equilibrium.

Plugging Eq.(6-6) into Eq.(6-5) yields:

$$c_t^e = \frac{1-\upsilon}{\gamma_\upsilon\sigma_C} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon\sigma_C} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t . \quad (6\text{-}7)$$

Plugging Eq.(6-7) into Eq.(6-1) yields:

$$\omega_t^e = \varphi n_t^e + \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t$$
$$= \frac{\varphi\gamma_\upsilon\sigma_C + 1-\upsilon}{\gamma_\upsilon \sigma_C} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t \quad . \text{(6-8)}$$

Plugging Eq.(6-7) into Eq.(6-4) yields:

$$s_t^e = \frac{1}{1-\upsilon}\left[\frac{1-\upsilon}{\gamma_\upsilon \sigma_C} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon \sigma_C} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t\right] - \frac{1}{1-\upsilon} z_t + \frac{1}{1-\upsilon} z_{2,t}^*$$
$$= \frac{1}{\gamma_\upsilon \sigma_C} n_t^e + \frac{1}{\gamma_\upsilon \sigma_C} a_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon \sigma_C} g_t \quad . \text{(6-9)}$$

Plugging Eqs. (6-8) and (6-9) into Eq.(6-2) yields:

$$
\begin{aligned}
mc_t &= \hat{\omega}_t + \upsilon \hat{s}_t + \omega_t^e + \upsilon s_t^e - a_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&= \hat{\omega}_t + \upsilon \hat{s}_t \\
&\quad + \frac{\varphi\gamma_\upsilon\sigma_C + 1 - \upsilon}{\gamma_\upsilon\sigma_C} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon\sigma_C} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \\
&\quad - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t \\
&\quad + \upsilon \left[\begin{array}{l} \frac{1}{\gamma_\upsilon\sigma_C} n_t^e + \frac{1}{\gamma_\upsilon\sigma_C} a_t + \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)-\gamma_\upsilon}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* \\ - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t \end{array} \right] \\
&\quad - a_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&= \hat{\omega}_t + \upsilon \hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau \\
&\quad + \frac{1+\varphi\gamma_\upsilon\sigma_C}{\gamma_\upsilon\sigma_C} n_t^e + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t + \frac{\upsilon\left[\eta(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\upsilon^2}{\gamma_\upsilon} z_{1,t}^* \\
&\quad - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{\upsilon\left[\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right]}{(1-\upsilon)\gamma_\upsilon} z_{2,t}^* - a_t - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t \\
&= \hat{\omega}_t + \upsilon \hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t \\
&\quad + \frac{1+\varphi\gamma_\upsilon\sigma_C}{\gamma_\upsilon\sigma_C} n_t^e + \frac{\upsilon\left\{\eta(2-\upsilon)(1-\upsilon)+\eta(2-\upsilon)-\gamma_\upsilon\right\}}{\gamma_\upsilon(1-\upsilon)} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* \\
&\quad - \frac{\upsilon\left\{\eta(2-\upsilon)+\eta\upsilon(2-\upsilon)-\gamma_\upsilon\right\}}{\gamma_\upsilon(1-\upsilon)} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t \\
&= \hat{\omega}_t + \upsilon \hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t + \frac{1+\varphi\gamma_\upsilon\sigma_C}{\gamma_\upsilon} n_t^e \\
&\quad + \frac{\upsilon\left\{\eta(2-\upsilon)\left[(1-\upsilon)+1\right]-\gamma_\upsilon\right\}}{\gamma_\upsilon(1-\upsilon)} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* \\
&\quad - \frac{\upsilon\left\{\eta(2-\upsilon)\left[(1-\upsilon)+1\right]-\gamma_\upsilon\right\}}{\gamma_\upsilon(1-\upsilon)} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t \\
&= \hat{\omega}_t + \upsilon \hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t + \frac{1+\varphi\gamma_\upsilon\sigma_C}{\gamma_\upsilon\sigma_C} n_t^e \\
&\quad + \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)} z_t - \frac{\upsilon}{\gamma_\upsilon} z_{1,t}^* - \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t
\end{aligned}
$$

.

Then, we have:

$$mc_t = \hat{\omega}_t + \upsilon\hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t + \frac{1+\varphi\gamma_\upsilon\sigma_C}{\gamma_\upsilon\sigma_C}n_t^e$$
$$+\frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_t - \frac{\upsilon}{\gamma_\upsilon}z_{1,t}^* - \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C}g_t \quad . \text{(6-15)}$$

Adding the LHS of Eq.(3-1-12) while subtracting the RHS of Eq.(3-1-12) into Eq.(6-15) yields:

$$mc_t = \hat{\omega}_t + \upsilon\hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau - a_t + \frac{1+\varphi\gamma_\upsilon\sigma_C}{\gamma_\upsilon}n_t^e + \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_t - \frac{\upsilon}{\gamma_\upsilon}z_{1,t}^*$$
$$-\frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]}{\gamma_\upsilon(1-\upsilon)}z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C}g_t$$
$$+\left[\left\{\hat{y}_t + y_t^e - \left[(\eta-1)\upsilon(2-\upsilon)+1\right]\left(\hat{s}_t + s_t^e\right) - (1-\upsilon)z_t - \upsilon z_{1,t}^* + (1-\upsilon)z_{2,t}^*\right\}\right]$$
$$= \hat{\omega}_t + \hat{y}_t + \left\{\upsilon - \left[(\eta-1)\upsilon(2-\upsilon)+1\right]\right\}\hat{s}_t + \frac{1}{1-\tau}\tau_t - \frac{1}{1-\tau}\tau + \frac{1+\varphi\gamma_\upsilon\sigma_C+\gamma_\upsilon}{\gamma_\upsilon}n_t^e$$
$$-\left[(\eta-1)\upsilon(2-\upsilon)+1\right]s_t^e$$
$$+\frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]-\gamma_\upsilon(1-\upsilon)^2}{\gamma_\upsilon(1-\upsilon)}z_t - \frac{\upsilon(1+\gamma_\upsilon)}{\gamma_\upsilon}z_{1,t}^* - \frac{\upsilon\left[\eta(2-\upsilon)^2-\gamma_\upsilon\right]-\gamma_\upsilon(1-\upsilon)^2}{\gamma_\upsilon(1-\upsilon)}z_{2,t}^*$$
$$-\frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C}g_t$$

(6-16)

Now, we turn to Eq.(6-1) $\mu_t^w = \omega_t - \varphi n_t - c_t$. Plugging Eq.(3-1-26) into Eq.(6-1) yields:

$$\mu_t^w = \omega_t - \varphi y_t - c_t + \varphi a_t$$
$$= \hat{\omega}_t - \varphi\hat{y}_t - \hat{c}_t + \omega_t^e - \varphi y_t^e - c_t^e + \varphi a_t \quad .$$

Plugging Eqs.(6-6), (6-7) and (6-8) into the previous expression yields:

$$
\begin{aligned}
\mu_t^w &= \hat{\omega}_t - \varphi\hat{y}_t - \hat{c}_t + \left[\begin{array}{l} \frac{\varphi\gamma_\upsilon\sigma_C + 1-\upsilon}{\gamma_\upsilon\sigma_C} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon\sigma_C} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* \\ - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t \end{array}\right] \\
&\quad -\varphi\left(n_t^e + a_t\right) \\
&\quad -\left[\frac{1-\upsilon}{\gamma_\upsilon\sigma_C} n_t^e + \frac{1-\upsilon}{\gamma_\upsilon\sigma_C} a_t + \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_t - \frac{\upsilon(1-\upsilon)}{\gamma_\upsilon} z_{1,t}^* - \frac{\eta\upsilon(2-\upsilon)}{\gamma_\upsilon} z_{2,t}^* - \frac{(1-\upsilon)\sigma_G}{\gamma_\upsilon\sigma_C} g_t\right] + \varphi a_t \\
&= \hat{\omega}_t - \varphi\hat{y}_t - \hat{c}_t + \frac{\varphi\gamma_\upsilon\sigma_C + 1-\upsilon}{\gamma_\upsilon\sigma_C} n_t^e - \frac{1-\upsilon}{\gamma_\upsilon\sigma_C} n_t^e - \varphi n_t^e \\
&= \hat{\omega}_t - \varphi\hat{y}_t - \hat{c}_t
\end{aligned}
$$

Then, we have:

$\mu_t^w = \hat{\omega}_t - \varphi\hat{y}_t - \hat{c}_t$. (6-14)